Measure Algebras and Applications, Year. 2025, Vol. 3, No. 1, pp. 104-112.

Measure Algebras and Applications

Generalization of certain well known polynomial inequalities

M. Shahmoradi[1], **M. Bidkham**[1]✉

1. Department of Mathematics, Semnan University, Semnan, Iran.

Email: m.shahmoradi@semnan.ac.ir,
mbidkham@semnan.ac.ir (Corresponding Author)

Article Info

Article history:
Received: 19 February 2025
Received in revised form:
24 May 2025
Accepted: 12 June 2025
Published Online:
25 June 2025

Keywords:
Complex polynomial,
Inequalities,
Maximum modulus,
Zeros

2020 Mathematics Subject Classification:
30A10, 30C10, 30C15

ABSTRACT

In this paper, we investigate a generalized class of complex polynomials of the form

$$G(z) = a_n z^n + \sum_{v=t}^{n} a_{n-v} z^{n-v}, 1 \leq t \leq n,$$

whose zeros are constrained to lie within or on the boundary of the disk $|z| \leq k$, $k \leq 1$. We establish new Bernstein-type inequalities for such polynomials, extending and improving earlier results due to Jain [6], Aziz and Rather [4]. Several known inequalities follow as special cases, and the results have implications for the growth estimates and derivative bounds of analytic functions within specified domains.

Cite this article: Shahmoradi, M., & Bidkham, M. (2025). Generalization of certain well known polynomial inequalities. *Measure Algebras and Applications*, 3(1), 104–112. http://doi.org/10.22091/maa.2026.14983.1051

Publisher: University of Qom
Doi:http://doi.org/10.22091/maa.2026.14983.1051

1. Introduction

Let $p(z)$ be a polynomial of degree n. It satisfies the following inequalities,

$$\max_{|z|=1}|p'(z)| \le n\max_{|z|=1}|p(z)|, \tag{1.1}$$

and

$$\max_{|z|=R}|p(z)| \le R^n\max_{|z|=1}|p(z)|, \quad R>1. \tag{1.2}$$

The inequality in (1.1) is widely recognized as Bernstein's inequality [5], while (1.2) is a straightforward consequence of the maximum modulus principle [9]. Equality in both cases occurs only when $p(z)$ is a constant multiple of z^n.

Restricting $p(z)$ to polynomials of degree n that do not vanish within $|z|<1$, the following inequalities hold,

$$\max_{|z|=1}|p'(z)| \le \frac{n}{2}\max_{|z|=1}|p(z)|, \tag{1.3}$$

and

$$\max_{|z|=R}|p(z)| \le \frac{R^n+1}{2}\max_{|z|=1}|p(z)|, \quad R>1. \tag{1.4}$$

Inequality (1.3), conjectured by Erdös, was later proven by Lax [7], while (1.4) is attributed to Ankeny and Rivlin [1].

Expanding on (1.3), Malik [8] demonstrated that if $p(z)$ does not vanish within $|z|<k$, where $k\ge 1$, then

$$\max_{|z|=1}|p'(z)| \le \frac{n}{1+k}\max_{|z|=1}|p(z)|. \tag{1.5}$$

Aziz and Dawood [3] refined (1.4), showing that if $p(z)\neq 0$ for $|z|<1$, then

$$\max_{|z|=R}|p(z)| \le \frac{R^n+1}{2}\max_{|z|=1}|p(z)| - \frac{R^n-1}{2}\min_{|z|=1}|p(z)|, \quad R>1. \tag{1.6}$$

For polynomials $p(z)$ of degree n with all zeros inside $|z|\le 1$, Aziz and Dawood [3] established the inequality

$$\max_{|z|=1}|p'(z)| \ge \frac{n}{2}\big(\max_{|z|=1}|p(z)| + \min_{|z|=1}|p(z)|\big). \tag{1.7}$$

Moreover, Malik [8] proved that if all zeros of $p(z)$ lie within $|z|<k$, where $k\le 1$, then

$$\max_{|z|=1}|p'(z)| \ge \frac{n}{1+k}\max_{|z|=1}|p(z)|. \tag{1.8}$$

Jain [6], introducing the parameter β, generalized (1.7) and (1.6) with the following results.

Theorem 1.1. *If $p(z)$ is a polynomial of degree n, with all zeros inside the closed unit disk, then for $|\beta|\le 1$,*

$$\max_{|z|=1}\Big|zp'(z)+\frac{n\beta}{2}p(z)\Big| \ge \frac{n}{2}\Big[(1+\mathrm{Re}(\beta))\max_{|z|=1}|p(z)| + \\ |(1+\mathrm{Re}(\beta))-|\beta||\min_{|z|=1}|p(z)|\Big]. \tag{1.9}$$

Theorem 1.2. *If $p(z)$ is a polynomial of degree n, with no zeros in $|z|<1$, then for every complex number $|\beta|\leq 1$, we have*

$$\max_{|z|=1}\Big|p(Rz)+\beta\Big(\frac{R+1}{2}\Big)^n p(z)\Big| \leq \frac{1}{2}\Big[\Big\{\Big|R^n+\beta\Big(\frac{R+1}{2}\Big)^n\Big| + \Big|1+\beta\Big(\frac{R+1}{2}\Big)^n\Big|\Big\}\max_{|z|=1}|p(z)| - \Big\{\Big|R^n+\beta\Big(\frac{R+1}{2}\Big)^n\Big| - \Big|1+\beta\Big(\frac{R+1}{2}\Big)^n\Big|\Big\}\min_{|z|=1}|p(z)|\Big]. \tag{1.10}$$

Aziz and Rather [4] extend (1.8), yielding,

Theorem 1.3. *For $p(z)$, a polynomial of degree n, with all zeros in $|z|\leq k$, where $k\leq 1$, and $|\beta|\leq 1$, we get*

$$\max_{|z|=1}\Big|zp'(z)+\frac{n\beta}{1+k}p(z)\Big| \geq \frac{n}{1+k}\{1+\mathrm{Re}(\beta)\}\max_{|z|=1}|p(z)|. \tag{1.11}$$

This paper focuses on improving and generalizing the inequalities in (1.7), (1.9), (1.11) and related results.

Theorem 1.4. *If $p(z)$ is a polynomial of degree n with all its zeros located in $|z|\leq k$, where $|k|\leq 1$ and $t-$fold zeros at the origin, then the following inequality holds,*

$$\max_{|z|=1}\Big|zp'(z)+\frac{n+tk}{1+k}\beta p(z)\Big| \geq \frac{n+tk}{1+k}\Big[\{1+\mathrm{Re}(\beta)\} \max_{|z|=1}|p(z)| + \frac{1}{k^{n+tk}}\{|1+k+\beta|-(1+\mathrm{Re}(\beta))\}\min_{|z|=k}|p(z)|\Big]. \tag{1.12}$$

Equality is achieved when $p(z)=(z+k)^n$ and $\beta\geq 0$. For $k=1$, $t=0$, we get the following improvement of the inequality (1.9).

Corollary 1.5. *If $p(z)$ is a polynomial of degree n with all zeros in $|z|\leq 1$, and for $|\beta|\leq 1$, then,*

$$\max_{|z|=1}\Big|zp'(z)+\frac{n\beta}{2}p(z)\Big| \geq \frac{n}{2}\Big[\{1+\mathrm{Re}(\beta)\}\max_{|z|=1}|p(z)| + \{|2+\beta|-(1+\mathrm{Re}(\beta))\}\min_{|z|=1}|p(z)|\Big]. \tag{1.13}$$

Remark 1.6. *Setting $\beta=0$ in Corollary 1.5 reduces inequality (1.13) to inequality (1.7).*

Remark 1.7. *Setting $t=0$ in Theorem 1.4 reduces inequality (1.12) to Theorem 1.4 in [10].*

Next, we present the following theorem, which generalizes Theorem 1.2.

Theorem 1.8. *If $p(z)$ is a polynomial of degree n with no zeros within $|z|<k$, where $k\geq 1$, with $t-$fold zeros at the origin, then for $|\beta|\leq 1$ and $R>1$, $R\geq r$, $rR\geq k^2$ and $|z|=1$, the inequality,*

$$\begin{aligned}&\max_{|z|=1}\Big|p(Rk^2z)+\beta\Big(\frac{R}{r}\Big)^t\Big(\frac{Rk+1}{rk+1}\Big)^{n-t}p(rk^2z)\Big|\\ &\leq \frac{1}{2}\Big[\Big\{k^{n+t}\Big|R^n+\beta\Big(\frac{R}{r}\Big)^t\Big(\frac{Rk+1}{rk+1}\Big)^{n-t}\Big| + \\ &\Big|1+\beta\Big(\frac{Rk+1}{rk+1}\Big)^{n-t}\Big(\frac{R}{r}\Big)^t\Big|\Big\}\max_{|z|=k}|p(z)|\\ &-\Big\{k^{n+t}\Big|R^n+\beta\Big(\frac{Rk+1}{rk+1}\Big)^{n-t}\Big(\frac{R}{r}\Big)^t\Big|\\ &-\Big|1+\beta\Big(\frac{Rk+1}{rk+1}\Big)^{n-t}\Big(\frac{R}{r}\Big)^t r^n\Big|\Big\}\min_{|z|=k}|p(z)|\Big],\end{aligned}$$

is valid. This result is also optimal, and equality holds for $p(z) = \lambda k^n + \gamma z^n$, *with* $|\lambda| \geq |\gamma|$.

Remark 1.9. *Setting* $k = 1$, $r = 1$, $t = 0$, *reduces Theorem 1.8 to Theorem 1.2, and if* $k = 1$, $r = 1$, $t = 0$, $\beta = 0$ *the inequality is simplified further to inequality (1.6).*

2. Lemmas

We will need the following lemmas to prove our theorems.

Lemma 2.1. *If* $p(z)$ *is a polynomial of degree* n *with all its zeros in* $|z| \leq k$, *where* $k \leq 1$, *with* $t-$*fold zeros at the origin, then for* $R \geq r$, $rR \geq k^2$,

$$|p(Rz)| \geq \left(\frac{R}{r}\right)^t \left(\frac{R+k}{r+k}\right)^{n-t} |p(rz)|, \quad \textit{for} \quad |z| = 1. \tag{2.1}$$

This lemma is due to Zargar [11].

For $t = 0$, $r = 1$, Lemma 2.1 reduces to a result due to Aziz [2].

Lemma 2.2. *If* $p(z)$ *is a polynomial of degree* n *with all its zeros in* $|z| \leq k$, *where* $k \leq 1$, *with* $t-$*fold zeros at the origin, then for* $|\beta| \leq 1$ *and* $|z| = 1$,

$$\left|zp'(z) + \left(\frac{n+tk}{1+k}\right)\beta p(z)\right| \geq \frac{n+tk}{1+k}\{1 + \mathrm{Re}(\beta)\}|p(z)|. \tag{2.2}$$

Proof. We know that if $p(z)$ is a polynomial of degree n with all zeros in $|z| \leq k$, where $k \leq 1$, with $t-$fold zeros at the origin [10], then

$$|zp'(z)| \geq \frac{n+tk}{1+k}|p(z)|.$$

Now by choosing an appropriate argument for β, we have,

$$\begin{aligned} &\left|zp'(z) + \frac{n+tk}{1+k}\beta p(z)\right| = |zp'(z)| + \left|\frac{n+tk}{1+k}\beta p(z)\right| \geq \\ &\frac{n+tk}{1+k}|p(z)| + \left|\frac{n+tk}{1+k}\beta p(z)\right| = \frac{n+tk}{1+k}\{1 + |\beta|\}|p(z)| \\ &\geq \frac{n+tk}{1+k}\{1 + \mathrm{Re}(\beta)\}|p(z)|. \end{aligned}$$

□

Lemma 2.3. *If* $p(z)$ *is a polynomial of degree* n *with all its zeros in* $|z| \leq k$, *where* $k \leq 1$, *with* $t-$*fold zeros at the origin, then for* $R \geq r$, $rR \geq k^2$, $|\beta| \leq 1$

$$\min_{|z|=1}\left|p'(z) + \frac{n+tk}{1+k}\beta p(z)\right| \geq \frac{n+tk}{k^{n+tk}}\left|1 + \frac{\beta}{1+k}\right| \min_{|z|=k}|p(z)|, \tag{2.3}$$

and

$$\begin{aligned} &\min_{|z|=1}\left|p(Rz) + \beta\left(\frac{R+k}{r+k}\right)^{n-t}\left(\frac{R}{r}\right)^t p(rt)\right| \geq \frac{1}{k^n}\left|R^n + \beta\left(\frac{R+k}{r+k}\right)^{n-t}\right. \\ &\left.\left(\frac{R}{r}\right)^t r^n\right| \min_{|z|=k}|p(z)|, \quad R > 1. \end{aligned} \tag{2.4}$$

Proof. If $p(z)$ has a zero on $|z| = k$, then inequalities become trivial. Assuming instead that $p(z)$ has all zeros within $|z| < k$, and letting $m = \min_{|z|=k}|p(z)|$, where $m > 0$, then for any complex α with $|\alpha| < 1$,

$$\left|\frac{\alpha m z^{n+tk}}{k^{n+tk}}\right| < |p(z)|, \quad \text{for} \quad |z| = k.$$

By Rouche's theorem, the polynomial $p(z) - \alpha m \left(\frac{z}{k}\right)^{n+tk}$, which has degree n, possesses all its zeros inside $|z| < k$ with t-fold zeros at the origin. From Lemma 2.2 (for $\beta = 0$), it follows that

$$\Big|zp'(z) - \frac{\alpha m(n+tk)}{k^{n+tk}} z^{n+tk}\Big| \geq \frac{n+tk}{1+k}\Big(p(z) - \alpha m\Big(\frac{z}{k}\Big)^{n+tk}\Big) \text{ for } |z| = 1.$$

Therefore, for $|\beta| < 1$, we obtain

$$zp'(z) - \frac{\alpha m(n+tk)}{k^{n+tk}} z^{n+tk} + \frac{n+tk}{1+k}\beta\Big(p(z) - \alpha m\Big(\frac{z}{k}\Big)^{n+tk}\Big) =$$
$$zp'(z) + \frac{n+tk}{1+k}\beta p(z) - \frac{\alpha m(n+tk)}{k^{n+tk}} z^{n+tk}\Big(1 + \frac{\beta}{1+k}\Big) \neq 0 \text{ for } |z| = 1.$$

Now, by appropriately choosing the argument of α and allowing $|\alpha| \longrightarrow 1$, we conclude that for $|z| = 1$ and $|\beta| < 1$

$$\Big|zp'(z) + \frac{n+tk}{1+k}\beta p(z)\Big| \geq \frac{n+tk}{k^{n+tk}}\Big|1 + \frac{\beta}{1+k}\Big|m.$$

Now by continuity on applying Lemma 2.1 to the polynomial $p(z) - \alpha m\left(\frac{z}{k}\right)^{n+tk}$ and using the same argument as above, the inequality (2.4) follows. □

Lemma 2.4. *If $p(z)$ is a polynomial of degree n, having no zeros in $|z| < k$ $(k \geq 1)$, except $t-$fold zeros at the origin, then for $|\beta| \leq 1$, $R \geq r$, $rR \geq k^2$ and $|z| = 1$,*

$$\Big|p(Rk^2z) + \beta\Big(\frac{R}{r}\Big)^t\Big(\frac{Rk+1}{1+kr}\Big)^{n-t}p(rk^2z)\Big| \leq$$
$$k^{n+t}\Big|Q(Rz) + \beta\Big(\frac{R}{r}\Big)^t\Big(\frac{Rk+1}{1+kr}\Big)^{n-t}Q(rz)\Big|, \tag{2.5}$$

where $Q(z) = z^{n+t}p\Big(\overline{\frac{1}{\bar{z}}}\Big)$.

Proof. Let $p(z) = z^t h(z)$. Then

$$Q(z) = z^{n+t}p\Big(\overline{\frac{1}{\bar{z}}}\Big) = z^t h\Big(\overline{\frac{1}{\bar{z}}}\Big) = z^t\Big(z^{n-t}h\Big(\overline{\frac{1}{\bar{z}}}\Big)\Big)$$

and by hypothesis, $p(z) \neq 0$ in $|z| < k$, $k \geq 1$, therefore, $Q(z)$ is the polynomial of degree n and has all its zeros in $|z| \leq \frac{1}{k}$ with t-fold zeros at the origin. As

$$|Q(z)| = \frac{1}{K^{n+t}}|p(k^2z)| \text{ for } |z| = \frac{1}{k},$$

the polynomial $k^{n+t}Q(z) - \alpha p(k^2z)$ of degree n has all its zeros in $|z| \leq \frac{1}{k}$ for $|\alpha| < 1$ and t-fold zeros at the origin. Therefore by Lemma 2.1 for $R > 1$ and $|z| = 1$, we have

$$|k^{n+t}Q(Rz) - \alpha p(Rk^2z)| \geq \Big(\frac{R}{r}\Big)^t\Big(\frac{Rk+1}{1+kr}\Big)^{n-t}|k^{n+t}Q(rz) - \alpha p(rk^2z)|.$$

It follows that for $|\beta| < 1$ and $|z| = 1$,

$$T(z) = k^{n+t}\Big\{Q(Rz) + \beta\Big(\frac{Rk+1}{1+rk}\Big)^{n-t}\Big(\frac{R}{r}\Big)^t Q(rz)\Big\}$$
$$-\alpha\Big\{p(Rk^2z) + \beta\Big(\frac{R}{r}\Big)^t\Big(\frac{Rk+1}{1+rk}\Big)^{n-t}p(rk^2z)\Big\} \neq 0. \tag{2.6}$$

This implies for $|\alpha| < 1$, $R > 1$ and $|z| = 1$,

$$\Big|p(Rk^2z) + \beta\Big(\frac{R}{r}\Big)^t\Big(\frac{Rk+1}{1+rk}\Big)^{n-t}p(rk^2z)\Big| \leq$$
$$k^{n+t}\Big|Q(Rz) + \beta\Big(\frac{R}{r}\Big)^t\Big(\frac{Rk+1}{1+rk}\Big)^{n-t}Q(rz)\Big|. \tag{2.7}$$

If inequality (2.7) is not true, then there exists a point $z = z_0$ with $|z_0| = 1$ such that

$$\Big|p(Rk^2z_0) + \beta\Big(\frac{Rk+1}{kr+1}\Big)^{n-t}\Big(\frac{R}{r}\Big)^t p(k^2rz_0)\Big| > k^{n+t}\Big|Q(Rz_0) + \beta\Big(\frac{Rk+1}{kr+1}\Big)^{n-t}\Big(\frac{R}{r}\Big)^t Q(rz_0)\Big|.$$

We have

$$\alpha = \frac{k^{n+t}\Big\{Q(Rz_0) + \beta\Big(\frac{Rk+1}{kr+1}\Big)^{n-t}\Big(\frac{R}{r}\Big)^t Q(rz_0)\Big\}}{p(Rk^2z_0) + \beta\Big(\frac{Rk+1}{kr+1}\Big)^{n-t} p(rk^2z_0)}.$$

Then $|\alpha| < 1$ and with this choice of α, we have from (2.6), $T(z_0) = 0$ for $|z_0| = 1$. But this contradicts the fact that $T(z) \neq 0$ for $|z| = 1$. This completes the proof of Lemma 2.4. □

Lemma 2.5. *If $p(z)$ is a polynomial of degree n, with $t-$fold zeros at the origin, then for $|\beta| \leq 1$, $R > r$, $rR \geq k^2$ and $|z| \geq 1$,*

$$\Big|p(Rz) + \beta\Big(\frac{R+k}{r+k}\Big)^{n-t}\Big(\frac{R}{r}\Big)^t p(rz)\Big| \leq \frac{|z^n|}{k^n}\Big|R^n + \beta\Big(\frac{R+k}{r+k}\Big)^{n-t}\Big(\frac{R}{r}\Big)^t r^n\Big| \max_{|z|=k\leq 1}|p(z)|. \tag{2.8}$$

Proof. Let $Q(z) = z^n p\Big(\frac{1}{\bar{z}}\Big)$ and $M = \max_{|z|=\frac{1}{k}\leq 1}|Q(z)|$. A direct application of Rouch's theorem shows that for every α with $|\alpha| > 1$, the polynomial $G(z) = Q(z) - \alpha M$ does not vanish in $|z| < \frac{1}{k}$ and $G^*(z) = z^n G\Big(\frac{1}{\bar{z}}\Big)$ has all it's zeros in $|z| \leq k$ with $t-$fold zeros at the origin. Applying Lemma 2.1 to the polynomial $G^*(z)$, we get for $|z| = 1$ and $R > 1$

$$|G^*(Rz)| \geq \Big(\frac{R}{r}\Big)^t\Big(\frac{R+k}{r+k}\Big)^{n-t}|G^*(rz)|. \tag{2.9}$$

Using Rouche's theorem again, it follows from (2.9) that for every β with $|\beta| < 1$ all the zeros of polynomial $G^*(Rz) + \beta\Big(\frac{R}{r}\Big)^t\Big(\frac{R+k}{r+k}\Big)^{n-t} G^*(rz)$ lie in $|z| < 1$.

Replacing $G^*(z)$ by $p(z) - \bar{\alpha}Mz^n$, we conclude that all the zeros of

$$S(z) = p(Rz) + \beta\Big(\frac{R+k}{r+k}\Big)^{n-t}\Big(\frac{R}{r}\Big)^t p(rz) - \bar{\alpha}\Big\{R^n z^n M + \beta\Big(\frac{R+k}{r+k}\Big)^{n-t}\Big(\frac{R}{r}\Big)^t M z^n r^n\Big\}$$

lie in $|z| < 1$ for $R > 1$, $k \leq 1$, $|\alpha| > 1$ and $|\beta| < 1$. This implies for $|\beta| < 1$, $R > 1$, $k \leq 1$ and $|z| \geq 1$,

$$\Big|p(Rz) + \beta\Big(\frac{R+k}{r+k}\Big)^{n-t}\Big(\frac{R}{r}\Big)^t p(rz)\Big| \leq |z^n|M\Big|R^n + \beta\Big(\frac{R+k}{r+k}\Big)^{n-t}\Big(\frac{R}{r}\Big)^t r^n\Big|.$$

Because if this is not true, then there is a point z_0 with $|z_0| \geq 1$, such that

$$\Big|p(Rz_0) + \beta\Big(\frac{R+k}{r+k}\Big)^{n-t}\Big(\frac{R}{r}\Big)^t p(rz_0)\Big| > |z_0^n|M\Big|R^n + \beta\Big(\frac{R+k}{r+k}\Big)^{n-t}\Big(\frac{R}{r}\Big)^t r^n\Big|.$$

We have

$$\bar{\alpha} = \frac{p(Rz_0) + \beta\Big(\frac{R+k}{r+k}\Big)^{n-t}\Big(\frac{R}{r}\Big)^t p(rz_0)}{z_0^n M\Big\{R^n + \beta\Big(\frac{R+k}{r+k}\Big)^{n-t}\Big(\frac{R}{r}\Big)^t r^n\Big\}},$$

then $|\alpha| > 1$ and with this choice of α, we have $S(z_0) = 0$ for $|z_0| \geq 1$. But this contradicts the fact that $S(z) \neq 0$ for $|z| \geq 1$. Thus for $|\beta| < 1$, $R > r$, $rR \geq k^2$ and $|z| \geq 1$,

$$\Big|p(Rz) + \beta\Big(\frac{R+k}{r+k}\Big)^{n-t}\Big(\frac{R}{r}\Big)^{t} p(rz)\Big| \leq |z^n| M \Big|R^n + \beta\Big(\frac{R+k}{r+k}\Big)^{n-t}\Big(\frac{R}{r}\Big)^{t} r^n\Big|,$$

replacing M by $\frac{1}{k^n}\max\limits_{|z|=k}|p(z)|$, we get inequality (2.8). □

Lemma 2.6. *If $p(z)$ is a polynomial of degree n, then for $|\beta| \leq 1$, $|z| = 1$ and $R > r$, $Rr \geq k^2$, $k \geq 1$ with $t-$fold zeros at the origin*

$$\begin{aligned}
&\Big|p(Rk^2 z) + \beta\Big(\frac{Rk+1}{rk+1}\Big)^{n-t}\Big(\frac{R}{r}\Big)^{t} p(rk^2 z)\Big| + k^{n+t}\Big|Q(Rz) + \beta\Big(\frac{R}{r}\Big)^{t}\Big(\frac{Rk+1}{rk+1}\Big)^{n-t} Q(rz)\Big| \\
&\leq \Big\{k^{n+t}\Big|R^n + \beta\Big(\frac{Rk+1}{rk+1}\Big)^{n-t}\Big(\frac{R}{r}\Big)^{t}\Big| + \Big|1 + \beta\Big(\frac{Rk+1}{rk+1}\Big)^{n-t}\Big(\frac{R}{r}\Big)^{t}\Big|\Big\} M, \qquad (2.10)
\end{aligned}$$

where $Q(z) = z^n p\Big(\frac{1}{\bar{z}}\Big)$, $M = \max\limits_{|z|=k}|p(z)|$.

Proof. If $M = \max\limits_{|z|=k\geq 1}|p(z)|$, then $|p(z)| \leq M$ for $|z| = k$. Therefore, for a given complex number λ with $|\lambda| > 1$, it follows by Rouche's theorem that the polynomial $w(z) = p(z) + \lambda M$ does not vanish in $|z| < k$. Hence from Lemma 2.4, for $|\beta| \leq 1$, $|z| = 1$ and $rR \geq k^2$, $R > r$, we get

$$\begin{aligned}
&\Big|p(Rk^2 z) + \beta\Big(\frac{R}{r}\Big)^{t}\Big(\frac{Rk+1}{rk+1}\Big)^{n-t} p(rk^2 z) + \lambda M\Big(1 + \beta\Big(\frac{R}{r}\Big)^{t}\Big(\frac{Rk+1}{rk+1}\Big)^{n-t}\Big)\Big| \leq \\
&k^{n+t}\Big|Q(Rz) + \bar{\lambda} M R^n z^n + \beta\Big(\frac{R}{r}\Big)^{t}\Big(\frac{Rk+1}{rk+1}\Big)^{n-t}\big(Q(rz) + \bar{\lambda} M z^n r^n\big)\Big|, \qquad (2.11)
\end{aligned}$$

choosing argument of λ in the right hand side of (2.11) such that

$$\begin{aligned}
&\Big|Q(Rz) + \beta\Big(\frac{Rk+1}{rk+1}\Big)^{n-t}\Big(\frac{R}{r}\Big)^{t} Q(rz) + \bar{\lambda} M z^n\Big\{R^n + \Big(\frac{R}{r}\Big)^{t}\Big(\frac{Rk+1}{rk+1}\Big)^{n-t} r^n\Big\}\Big| \\
&= M|\lambda||z^n|\Big|R^n + \beta\Big(\frac{Rk+1}{rk+1}\Big)^{n-t}\Big(\frac{R}{r}\Big)^{t} r^n\Big| - \Big|Q(Rz) + \beta\Big(\frac{Rk+1}{rk+1}\Big)^{n-t}\Big(\frac{R}{r}\Big)^{t} Q(rz)\Big|,
\end{aligned}$$

by Lemma 2.5, we get

$$\begin{aligned}
&\Big|p(Rk^2 z) + \beta\Big(\frac{Rk+1}{rk+1}\Big)^{n-t}\Big(\frac{R}{r}\Big)^{t} p(rk^2 z)\Big| - M|\lambda|\Big|1 + \beta\Big(\frac{Rk+1}{rk+1}\Big)^{n-t}\Big(\frac{R}{r}\Big)^{t}\Big| \leq \\
&M k^{n+t}|\lambda||z^n|\Big|R^n + \beta\Big(\frac{Rk+1}{rk+1}\Big)^{n-t}\Big(\frac{R}{r}\Big)^{t}\Big| - k^{n+t}\Big|Q(Rz) + \beta\Big(\frac{Rk+1}{rk+1}\Big)^{n-t}\Big(\frac{R}{r}\Big)^{t} Q(rz)\Big|.
\end{aligned}$$

Equivalently

$$\begin{aligned}
&\Big|p(Rk^2 z) + \beta\Big(\frac{Rk+1}{rk+1}\Big)^{n-t}\Big(\frac{R}{r}\Big)^{t} p(rk^2 z)\Big| \\
&+ k^{n+t}\Big|Q(Rz) + \beta\Big(\frac{Rk+1}{rk+1}\Big)^{n-t}\Big(\frac{R}{r}\Big)^{t} Q(rz) \\
&\leq M|\lambda|\Big\{k^{n+t}\Big|R^n + \beta\Big(\frac{Rk+1}{rk+1}\Big)^{n-t}\Big(\frac{R}{r}\Big)^{t}\Big| + \Big|1 + \beta\Big(\frac{Rk+1}{rk+1}\Big)^{n-t}\Big(\frac{R}{r}\Big)^{t}\Big|\Big\}.
\end{aligned}$$

Finally letting $|\lambda| \longrightarrow 1$, we get inequality (2.10) and this completes the proof of Lemma 2.6. □

3. Proof of the theorems

Proof of Theorem 1.4. If $p(z)$ has a zero on $|z| = k$, inequality (1.12) reduces to

$$\max_{|z|=1}\Big|zp'(z) + \frac{n+tk}{1+k}\beta p(z)\Big| \geq \frac{n+tk}{1+k}\{(1 + \mathrm{Re}(\beta)\}\max_{|z|=1}\Big|p(z)\Big|, \qquad (3.1)$$

which is straightly followed from Lemma 2.2. So we suppose that $p(z)$ has all its zeros in $|z| < k$, then $m = \min\limits_{|z|=k}|p(z)| > 0$. Therefore if α is a complex number such that $|\alpha| < 1$, then it follows by Rouche's theorem that the polynomial $p(z)-\alpha m\left(\frac{z}{k}\right)^{n+tk}$ has all its zeros in $|z|<k$, $k\leq 1$, with $t-$fold zeros at the origin.

Applying Lemma 2.2, we get for $|z|=1$

$$\left|zp'(z)-\alpha m(n+tk)\Big(\frac{z}{k}\Big)^{n+tk}+\frac{n+tk}{1+k}\beta\Big(p(z)-\alpha m\Big(\frac{z}{k}\Big)^{n+tk}\Big)\right|\geq$$

$$\frac{n+tk}{1+k}(1+\mathrm{Re}(\beta))\Big|p(z)-\alpha m\Big(\frac{z}{k}\Big)^{n+tk}\Big|. \tag{3.2}$$

Equivalently

$$\left|zp'(z)+\frac{n+tk}{1+k}\beta p(z)-\alpha m(n+tk)\Big(\frac{z}{k}\Big)^{n+tk}\Big(1+\frac{\beta}{1+k}\Big)\right|$$
$$\geq\frac{n+tk}{1+k}(1+\mathrm{Re}(\beta))\Big|p(z)-\alpha m\Big(\frac{z}{k}\Big)^{n+tk}\Big|.$$

By Lemma 2.3 (inequality (2.3)), we have

$$\min_{|z|=1}\Big|p'(z)+\frac{n+tk}{1+k}\beta p(z)\Big|\geq\frac{n+tk}{k^{n+tk}}\Big|1+\frac{\beta}{1+k}\Big|\min_{|z|=k}|p(z)|.$$

By choosing α suitable argument of α and letting $|\alpha|\longrightarrow 1$, for $|z|=1$, we get

$$\Big|zp'(z)+\frac{n+tk}{1+k}\beta p(z)\Big|-\frac{m(n+tk)}{k^{n+tk}}\ \Big|1+\frac{\beta}{1+k}\Big|\geq$$

$$\frac{n+tk}{1+k}(1+\mathrm{Re}(\beta))\Big\{|P(Z)|-\frac{m}{k^{n+tk}}\Big\}. \tag{3.3}$$

From which we can obtain Theorem 1.4.

Proof of Theorem 1.8. If $p(z)$ has a zero on $|z|=k$, then by Lemma 2.4, we have for $|z|=1$,

$$\Big|p(Rk^2z)+\beta\Big(\frac{R}{r}\Big)^t\Big(\frac{Rk+1}{rk+1}\Big)^{n-t}p(rk^2z)\Big|\leq$$
$$k^{n+t}\Big|Q(Rz)+\beta\Big(\frac{R}{r}\Big)^t\Big(\frac{Rk+1}{rk+1}\Big)^{n-t}Q(rz)\Big|.$$

On applying Lemma 2.6, we get the conclusion of Theorem 1.8. Therefore we assume that $p(z)$ has all its zeros in $|z|>k$ with $t-$fold zeros at the origin. Let $m=\min\limits_{|z|=k}|p(z)|$, then $m\leq|p(z)|$, for $|z|=k$. If α is a complex number such that $|\alpha|<1$, then it follows from Rouche's theorem that the polynomial $p_1(z)=p(z)-\alpha m$ having no zeros in $|z|<k$. Hence by Lemma 2.4, we get for $|z|<1$

$$\Big|p_1(Rk^2z)+\beta\Big(\frac{R}{r}\Big)^t\Big(\frac{Rk+1}{rk+1}\Big)^{n-t}p_1(rk^2z)\Big|\leq k^{n+t}$$
$$\Big|Q_1(Rz)+\beta\Big(\frac{R}{r}\Big)^t\Big(\frac{Rk+1}{rk+1}\Big)^{n-t}Q_1(rz)\Big|,$$

where $Q_1(z)=z^{n+t}p\Big(\frac{1}{\bar z}\Big)-z^{n+t}\alpha m$. Equivalently

$$\left|\Big\{p(Rk^2z)+\beta\Big(\frac{Rk+1}{rk+1}\Big)^{n-t}\Big(\frac{R}{r}\Big)^t p(k^2rz)\Big\}-\alpha m\Big\{1+\beta\Big(\frac{R}{r}\Big)^t\Big(\frac{Rk+1}{rk+1}\Big)^{n-t}\Big\}\right|\leq$$
$$\left|k^{n+t}\Big\{Q(Rz)+\beta\Big(\frac{Rk+1}{rk+1}\Big)^{n-t}\Big(\frac{R}{r}\Big)^t Q(rz)\Big\}-mk^{n+t}\alpha z^n\Big\{R^n+\beta\Big(\frac{Rk+1}{rk+1}\Big)^{n-t}\Big(\frac{R}{r}\Big)^t r^n\Big\}\right|,$$

or for $|z|=1$

$$\Big|p(Rk^2z)+\beta\Big(\frac{Rk+1}{rk+1}\Big)^{n-t}\Big(\frac{R}{r}\Big)^t p(k^2rz)\Big|-|\alpha|m\Big|1+\beta\Big(\frac{R}{r}\Big)^t\Big(\frac{Rk+1}{rk+1}\Big)^{n-t}\Big|\leq$$

$$\left|k^{n+t}\Big|Q(Rz)+\beta\Big(\frac{Rk+1}{rk+1}\Big)^{n-t}\Big(\frac{R}{r}\Big)^t Q(rz)\Big|-mk^{n+t}|\alpha|\Big|R^n+\beta\Big(\frac{Rk+1}{rk+1}\Big)^{n-t}\Big(\frac{R}{r}\Big)^t r^n\Big|\right|. \tag{3.4}$$

On applying lemma 2.3 (inequality (2.4)) to the polynomial Q(z) and rewriting (3.4), we get for $|z| = 1$

$$\left|p(Rk^2z) + \beta\Big(\frac{Rk+1}{rk+1}\Big)^{n-t}\Big(\frac{R}{r}\Big)^t p(rk^2z)\right| - |\alpha|m\left|1 + \beta\Big(\frac{Rk+1}{rk+1}\Big)^{n-t}\Big(\frac{R}{r}\Big)^t\right| \leq$$

$$k^{n+t}\left|Q(Rz) + \beta\Big(\frac{Rk+1}{rk+1}\Big)^{n-t}\Big(\frac{R}{r}\Big)^t Q(rz)\right| - mk^{n+t}|\alpha|\left|R^n + \beta\Big(\frac{Rk+1}{rk+1}\Big)^{n-t}\Big(\frac{R}{r}\Big)^t r^n\right|.$$

As $|\alpha| \longrightarrow 1$, we obtain for $|z| = 1$

$$\left|p(Rk^2z) + \beta\Big(\frac{Rk+1}{rk+1}\Big)^{n-t}\Big(\frac{R}{r}\Big)^t p(rk^2z)\right| - k^{n+t}\left|Q(Rz) + \beta\Big(\frac{Rk+1}{rk+1}\Big)^{n-t}\Big(\frac{R}{r}\Big)^t Q(rz)\right|$$

$$\leq m\left\{\left|1 + \beta\Big(\frac{Rk+1}{rk+1}\Big)^{n-t}\Big(\frac{R}{r}\Big)^t\right| - k^{n+t}\left|R^n + \beta\Big(\frac{Rk+1}{rk+1}\Big)^{n-t}\Big(\frac{R}{r}\Big)^t r^n\right|\right\}.$$

Now by Lemma 2.6, we get the derived result and this completes the proof of Theorem 1.8.

References

[1] Ankeny, N.C., & Rivlin, T.J. (1955). On a theorem of S.Bernstein. *Pacific J. Math*, 5, 849–851.

[2] Aziz, A. (1987). Growth of polynomials whose zeros are within or outside a circle. *Bulletin of the Australian Mathematical Society*, 35(2), 247–256.

[3] Aziz, A., & Dawood, Q.M. (1988). Inequalities for a polynomial and its derivative. *Journal of Approximation Theory*, 54(3), 306–313.

[4] Aziz, A., & Rather, N.A. (2004). Some compact generalizations of Bernstein-type inequalities for polynomials. *Mathematical Inequalities and Applications*, 7(3), 393–403.

[5] Bernstein, S. (1930). Sur la limitation des dérivées des polynômes. *C. R. Acad. Sc. Paris*, 190, 338–340.

[6] Jain, V.K. (2000). Inequalities for a polynomial and its derivative. *Proc. Indian Acad. Sci*, 110, 137–146.

[7] Lax, P.D. (1944). Proof of a conjecture of P.Erdos on the derivative of a polynomial. *Bull. Amer. Math. Soc*, 50, 509–513.

[8] Malik, M.A. (1969). On the Derivative of a Polynomial. *Journal of the London Mathematical Society*, 1, 57–60.

[9] Polya, G., & Szego, G. (1976). Problems and Theorems in Analysis. *Springer-Verlag.*

[10] Soleiman Mezerji, H.A., Bidkham, M., & Zireh, A. (2012). Bernstien type inequalities for polynomial and its derivative. *Journal of Advanced Research in Pure Mathematics*, 4, 26–33.

[11] Zargar, B.A. (2014). On an inequality of Paul Turan. *Matematički Vesnik*, 66(2), 148–154.