Measure Algebras and Applications, Year. 2025, Vol. 3, No. 1, pp. 93-103.

Measure Algebras and Applications

Bipositive isomorphisms on semigroup algebras

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Article Info

Article history:
Received: 4 February 2025
Received in revised form:
22 April 2025
Accepted: 14 May 2025
Published Online:
25 June 2025

Keywords:
Bipositive isomorphisms,
Foundation semigroups,
Semigroup algebras

2020 Mathematics Subject Classification:
43A10, 22015

ABSTRACT

Let S be a locally compact foundation semigroup with identity and $M_a(S)$ be its semigroup algebra. In the present article, we show that if T is a bipositive isomorphism from $M_a(S_1)$ onto $M_a(S_2)$, then T is an isometry and S_1 and S_2 are isomorphic locally compact semigroups. Indeed, we have obtained a generalization of a well-known result of Wendel [9] and Kawada [6] for locally compact groups to a more general setting of locally compact foundation semigroups. Also we show that if T is a bipositive isomorphism from $M_a(S_1)^{**}$ onto $M_a(S_2)^{**}$, then S_1 and S_2 are isomorphic locally compact semigroups.

Cite this article: Babagolzadeh, M., Mohammadzadeh, B., & Motiee, M. (2025). Bipositive isomorphisms on semigroup algebras. *Measure Algebras and Applications*, 3(1), 93–103. http://doi.org/10.22091/maa.2026.14931.1048

 Publisher: University of Qom
Doi:http://doi.org/10.22091/maa.2026.14931.1048

1. Preliminaries

Throughout this paper, S denotes a locally compact Hausdorff topological semigroup. Let S be a locally compact Hausdorff topological foundation semigroup. The space of all bounded complex regular Borel measures on S is denoted by $M(S)$. This space with the convolution multiplication $*$ and the total variation norm defines a Banach algebra. The space of all measures $\mu \in M(S)$ for which that mappings $x \mapsto \delta_x * |\mu|$ and $x \mapsto |\mu| * \delta_x$ from S into $M(S)$ are weakly continuous, is denoted by $M_a(S)$, where δ_x denotes the Dirac measure at x. The locally compact semigroup S is called a foundation semigroup if S coincides with the closure of the set $\cup\{\text{supp}(\mu) : \mu \in M_a(S)\}$.

Note that if S is a foundation semigroup with identity, then $M_a(S)$ has a bounded approximate identity, see [7]. Recall that $M_a(S)$ is a two-sided closed L-ideal of $M(S)$ from [7], with the norm

$$\|\mu\| = \int_s d|\mu|$$

and the convolution product

$$\int_S f(x)d(\mu * \nu)(x) = \int_S \int_S f(xy)d\mu(x)d\nu(y).$$

Let us point out that the second dual $M_a(S)^{**}$ of $M_a(S)$ is a Banach algebra with the first Arens product $\odot$ defined by the equations

$$(F \odot H)(f) = F(Hf)$$

$$(Hf)(\mu) = H(f\mu)$$

$$(f\mu)(\nu) = f(\mu * \nu)$$

for all $F, H \in M_a(S)^{**}$, $f \in M_a(S)^*$, and $\mu, \nu \in M_a(S)$.

Let $LUC(S)$ be the set of all left uniformly continuous functions on S; recall that a function $g \in C_b(S)$ is called left uniformly continuous if the mapping $x \mapsto_x g$ from S into $C_b(S)$ is continuous, where $C_b(S)$ denotes the space of all bounded continuous complex-valued functions on S; as usual $C_0(S)$ denotes the space of functions in $C_b(S)$ vanishing at infinity and $C_c(S)$ denotes its subspace of functions with compact support.

The set of all complex-valued bounded functions g on S that are $M_a(S)$-measurable; that is, μ-measurable for all $\mu \in M_a(S)$ is denoted by $L^\infty(S, M_a(S))$. For every $g \in L^\infty(S, M_a(S))$, define

$$\|g\|_\infty = \sup\{\|g\|_{\infty,|\mu|} : \mu \in M_a(S)\},$$

where $\|.\|_{\infty,|\mu|}$ denotes the essential supremum norm with respect to $|\mu|$. Observe, $L^\infty(S, M_a(S))$ with the complex conjugation as involution, the pointwise operations and the norm $\|.\|_\infty$ is a commutative C^*-algebra. The duality

$$\tau(g)(\mu) := \mu(g) = \int_S g d\mu$$

defines a linear mapping τ from $L^\infty(S, M_a(S))$ into $M_a(S)^*$. It is well-known that if S is a foundation semigroup with identity, then τ is an isometric isomorphism of $L^\infty(S, M_a(S))$ onto $M_a(S)^*$.

Let X be a subspace of $L^\infty(S, M_a(S))$ which is left and right translations invariant; that is, ${}_sg$ and g_s are in X for all $g \in X$ and $s \in S$, where

$$({}_sg)(t) = g(st) \text{ and } (g_s)(t) = g(ts)$$

for all $t \in S$.

Several interesting isomorphism theorems have already been proved for various convolution algebras over groups G_i for $(i = 1, 2)$. First Kawada in [6] showed that if we have a bipositive algebra isomorphism between group algebras, then the underlying locally compact groups must be isomorphic. In [4], H. Farhadi proved similar results to Kawada's result for other Banach algebras related to locally compact groups by first showing that any such bipositive algebra isomorphism must in fact be an isometric isomorphism.

In this paper, as a generalization of these theorems, we show that the existence of a bipositive isomorphism between semigroup algebras and their second duals implies that their underlying locally compact semigroups must be isomorphic.

2. Bipositive isomorphisms of semigroup algebras

In this section, we prove similar results to Wendel's result for semigroup algebras and we show that the existence of a bipositive algebra isomorphism between the measure algebras $M(S_1)$ and $M(S_2)$ implies that their underlying locally compact semigroups must be isomorphic. Also we give a complete description of bipositive algebra isomorphisms between semigroup algebras.

Definition 2.1. *Let S be a locally compact semigroup. Then*

(1) The function $f \in C_0(S)$ is called positive if for every $x \in S$, $f(x) \geq 0$;
(2) The measure $\mu \in M(S)$ is called positive if for every positive $f \in C_0(S)$, $\langle \mu, f\rangle \geq 0$;
(3) The function $f \in LUC(S)$ is called positive if $f(x) \geq 0$, for all $x \in S$;
(4) The functional $m \in LUC(S)^*$ is positive if $\langle m, f\rangle \geq 0$, for every positive function f in $LUC(S)$;
(5) Let A and B be ordered vector spaces. An operator $T : A \longrightarrow B$ is called positive if for each positive element $a \in A$, $T(a) \geq 0$ in B. The operator T is called bipositive if T is a bijection and both T and T^{-1} are positive operators.

In the following theorem, we prove Wendel's result for semigroup algebras.

Theorem 2.2. *Let S_1 and S_2 be two locally compact semigroups and T be a bipositive isomorphism of $M_a(S_1)$ onto $M_a(S_2)$. Then T is an isometry.*

Proof. T and T^{-1} are both order-preserving operators

$$\mu \leq \nu, \qquad T(\mu) \leq T(\nu), \qquad \mu, \nu \in M(S_1)$$

and therefore are bounded. So the ratio $\|T\mu\|/\|\mu\|$ is bounded as $\mu \neq 0, \infty$, over $M_a(S_1)$. If $\mu \in M_a(S_1)^+$ it follows by induction that $\|\mu^n\| = \|\mu\|^n$.

Since μ is positive and T is bipositive so $T\mu$ is also positive and $T(\mu^n) = (T\mu)^n$, it follows that for fixed positive $\mu \neq 0$ the retio $(\|T\mu\|/\|\mu\|)^n$ is bounded above and below for $n \geq 0$. Consequently T is isometric at least for the positive elements of $M_a(S_1)$.

Now if $\mu \in M(S_1)$ and $\mu = \mu^+ + \mu^-$, where μ^+ and μ^- denote respectively the positive and negative parts of μ, then

$$\begin{aligned}\|\mu\| &= \|\mu^+ + \mu^-\| \\ &= \|\mu^+\| + \|\mu^-\| \\ &= \|T\mu^+\| + \|T\mu^-\| \geq \|T\mu^+ + T\mu^-\| \\ &= \|T\mu\|.\end{aligned}$$

With a similar argument for T^{-1}, we obtain the result

$$\|\mu\| = \|T^{-1}T\mu\| \leq \|T\mu\| \leq \|\mu\|,$$

then T is an isometry. □

Lemma 2.3. *Let S be a locally compact semigroup. Then a left multiplier $L_\mu : \nu \longrightarrow \mu * \nu$ on $M_a(S)$ is positive if only if μ is positive.*

Proof. If $\mu \in M(S)$ is positive, then it is clear that left multiplier $L_\mu(\nu) = \mu * \nu$ is positive. Now, we consider the case that L_μ is positive, by the above conclusion, we have $L_\mu(\nu_i) \longrightarrow \mu$ where (ν_i) is a net bounded approximate identity for $M_a(S)$. Finally, as (ν_i) is positive, so $L_\mu(\nu_i)$ is positive and consequently μ is positive. □

Recall that a locally compact semigroup S is said to be left compactly cancellative if $C^{-1}D$ is a compact subset of S for all compact subsets C and D of S, where

$$C^{-1}D = \{x \in S : cx \in D \text{ for some } c \in C\}.$$

Right compactly cancellative locally compact semigroups are defined similarly. A semigroup which is both left and right compactly cancelletive is called compactly cancellative.

Suppose that A is a Banach algebra, for any $a, b \in A$, we define

$$L(ab) = L(a)b$$

$$R(ab) = aR(b)$$

where operators L and R are called multipliers, also $\mathcal{M}(A)$ denotes the set of multipliers on A.

Theorem 2.4. *Suppose that S is a locally compact cancellative foundation semigroup with identity. Then the left multiplier algebra of $M_a(S)$ is bipositively and algebraically isomorphic to $M(S)$.*

Proof. Suppose that $T : M(S) \longrightarrow \mathcal{M}(M_a(S))$ such that $T(\mu) = L_\mu$. From [8], we know that if S is a compactly cancellative foundation semigroup with identity, thus the multiplier algebra of $M_a(S)$ is isomorphic with $M(S)$.

We recall that if T is a left multiplier on $M_a(S)$ and let a net $\{\nu_\alpha\}$ from $M_a(S)$, then the set $\{T(\nu_\alpha)\}$ is a bounded subnet of $M(S)$. So there exists a subnet $\{\nu_{\alpha_i}\}$ of $\{\nu_\alpha\}$ and $\mu \in M(S)$ for which that

$$T(\nu_{\alpha_i}) \xrightarrow{w^*} \mu.$$

Now if $\nu \in M_a(S)$, then

$$T(\nu * \nu_{\alpha_i}) \xrightarrow{w^*} T(\nu), \qquad T(\nu * \nu_{\alpha_i}) = T(\nu_{\alpha_i}) * \nu \xrightarrow{w^*} \mu * \nu$$

consequently $T(\nu) = \nu * \mu$, we have that T is algebra isomorphism. Finally, Lemma 2.3 proves that T is bipositive. □

Remark 2.5. *Let S be a locally compact foundation semigroup with identity e. The right annihilator of $M_a(S)$ in $M(S)$ is zero.*

Proof. Let μ_α be a bounded approximate identity for $M_a(S)$. We know that $\mu_\alpha \to \delta_e$ in ω^*-topology. Thus

$$\mu = \delta_e * \mu = \omega^* - \lim(\mu_\alpha * \mu) = 0.$$

□

In the following theorem, we give a generalization of Kawada's result in [6], for locally compact semigroups with identity.

Theorem 2.6. *Let S_1 and S_2 be locally compact semigroups and T be a bipositive algebra isomorphism from $M(S_1)$ onto $M(S_2)$. Then the locally compact semigroups S_1 and S_2 are isomorphic, and also there exists a continuous character $\chi : S_1 \to \mathbb{T}$ where $\mathbb{T}$ is the circle semigroup and an isomorphism $\psi : S_1 \to S_2$ of the locally compact semigroups S_1 and S_2 such that for each $x \in S_1$, we have*

$$T(\delta_x) = \int_\psi \chi(t) d(\delta_x(t)) = \chi(x)\delta_{\phi(x)}.$$

Proof. Suppose that $T : M(S_1) \longrightarrow M(S_2)$ is a bipositive algebra isomorphism. By [1], Theorem 4.3, T and T^{-1} are bounded operators. Suppose that x in S_1 is given. Since δ_x is a positive measure and T is a positive operator, $T(\delta_x)$ is a positive measure. Also since T is an algebra isomorphism, it preserves the extreme points of the unit ball, by [3], Theorem V.8.4, for each $x \in S_1$, there exist an isomorphism $\psi(x) : S_1 \to S_2$ and a continuous character $\chi(x) : S_1 \to \mathbb{T}$ such that

$$T(\delta_x) = \chi(x)\delta_{\psi(x)}. \tag{2.1}$$

Since T is an algebra isomorphism and multiplicative and $\delta_x * \delta_y = \delta_{xy}$ for each $x, y \in S_1$, we can see that $\psi : S \longrightarrow S_2$ and $\chi : S_1 \longrightarrow \mathbb{T}$ via (2.1) are multiplicative,

$$T(\delta_{xy}) = T(\delta_x * \delta_y) = T(\delta_x) * T(\delta_y).$$

Suppose that (x_α) is a net in S_1 that is convergent to e_{s_1}, the identity element of S_1. The map $S_1 \mapsto M(S_1) : x \mapsto \delta_x$ is strong operator continuous, for every $\mu \in M_a(S_1)$,

$$\delta_{x_\alpha} * \mu \xrightarrow{\|\cdot\|} \mu. \tag{2.2}$$

Since T is bounded, we have

$$T(\delta_{x_\alpha} * \mu) \xrightarrow{\|\cdot\|} T(\mu),$$

in $M(S_2)$. Hence

$$T(\delta_{x_\alpha}) * T(\mu) \xrightarrow{\|\cdot\|} T(\mu). \tag{2.3}$$

Let U be a precompact neighborhood of e_{s_1}. Without loss of generality we can assume $x_i \in U$, for all i. Then

$$\|T(\delta_{x_\alpha})\| \leq \|T\|\|\delta_{x_\alpha}\| \leq \|T\|.$$

Hence, the net $(T(\delta_{x_\alpha}))$ is bounded in $M(S_2)$, and so it has a subnet $(T(\delta_{x_{\alpha_i}}))$ converging weak-star to some $\mu \in M(S_2)$. Then by (2.3), we have that

$$\nu * T(\mu) = T(\psi).$$

Applying T^{-1} to the two sides of this equation yields

$$T^{-1}(\nu) * \mu = \psi.$$

Hence by Remark 2.5, $T^{-1}(\nu) = \delta_{e_{s_1}}$, or equivalently $T(\delta_{e_{s_1}}) = \nu$. Hence

$$\nu = \chi(e_{s_1})(\delta_{\psi(e_{s_1})}) = \chi(e_{s_1})(\delta_{e_{s_2}}). \tag{2.4}$$

The equation (2.4), in particular, shows that $\chi(x_i) \to \chi(e_{s_1})$ and $\psi(x_i) \to \psi(e_{s_1})$. Hence, ψ and χ are continuous. To prove that ψ is a bijection, we note that corresponding to T^{-1}, there exist $\beta : S_2 \to (0, \infty)$ and $\phi : S_2 \to S_1$ such that

$$T^{-1}(\delta_y) = \beta(y)\delta_{\phi(y)} \ (y \in H).$$

It follows from the equations $T(T^{-1}(\delta_y)) = \delta_y$ and $T^{-1}(T(\delta_x)) = \delta_x$ that ϕ is a bijection, $\phi = \psi^{-1}$ and $\beta(\psi(x)) = 1 \setminus \chi(x)$ that $\chi(x) \neq 0$, for all $x \in S_1$. By symmetry, ϕ is continuous. Therefore ψ is an isomorphism of topological semigroups from S_1 onto S_2.

Now, we show that for every $x \in S_1$ we have

$$\|T^{-1}\|^{-1} \leq \chi\|\delta_{\psi(x)}\| \leq \|T\|. \tag{2.5}$$

Since T is a bounded operator, for all $x \in S_1$, we have

$$\chi(x) = \chi(x)\|\delta_{\psi(x)}\| = \|T(\delta(x))\| \leq \|T\|\|\delta(x)\| = \|T\|.$$

A similar argument using T^{-1} shows that

$$\|T^{-1}\|^{-1} \leq \chi(x)\|\delta_{\psi(x)}\|.$$

Thus we have established the inequalities in (2.5). □

Theorem 2.7. *Let T be a bipositive algebra isomorphism from $M(S_1)$ onto $M(S_2)$. Define the mapping $K_{\chi,\psi} : C_0(S_2) \to C_0(S_1)$ where $K_{\chi,\psi}(f) = \chi.f \circ \psi$. Then $T = K^*_{\chi,\psi}$.*

Proof. By Theorem 2.6, there exist an isomorphism ψ of locally compact semigroups S_1 onto S_2, a continuous character $\chi : S_1 \to \mathbb{T}$ and positive constants M and m such that

$$T(\delta_x) = \chi(x)\delta_{\phi(x)} \quad (m \leq \chi(x) \leq M). \tag{2.6}$$

For each $x \in S_1$, letting $\mu \in M(S_1)$ be such that $\|\mu\| = 1$, it suffices to show that $T(\mu) = K^*_{\chi,\psi}(\mu)$, where the dual mapping $T_{\chi,\psi} = K^*_{\chi,\psi}$ from $M(S_1)$ onto $M(S_2)$ is also a bounded bipositive linear isomorphism. By Theorem 2.6, $T_{\chi,\psi}(\delta_x) = \chi(x)\delta_\psi$ is multiplicative on point masses on $M(S_1)$. We note that the linear span of point masses is weak-star dense in $M(S_1)$, the convolution product is separately weak-star continuous and $T_{\chi,\psi} = K^*_{\chi,\psi}$ is weak-star continuous. Clearly, $T_{\chi,\psi}$ is invertible with $T_{\chi,\psi^{-1}} = T_{\beta,\psi^{-1}}$, where $\beta = 1 \setminus \chi \circ \psi^{-1}$. Therefore $T_{\chi,\psi}$ is a bipositive algebra isomorphism. Taking μ in $M(S_1)$ with norm 1, we can find a net (μ_β) in $M(S_1)$ such that

$$\lim \|\mu_\beta * \nu - \mu * \nu\| = 0 \ (\nu \in M_a(S_1)). \tag{2.7}$$

By (2.6), $T(\mu_\beta) = K^*_{\chi,\psi}(\mu_\beta)$ for each β. We claim that $T(\mu_\beta) \overset{w^*}{\to} T(\mu)$ in $M(S_2)$. To see this let μ' be a weak-star limit point of $T(\mu_\beta)$ in $M(S_2)$ and let $\mu_{\beta(i)}$ be a subnet of μ_β such that $T(\mu_{\beta(i)}) \overset{w^*}{\to} \mu'$. Observe that it suffices now to show that $\mu' = T(\mu)$ to simplify notation, we can assume that $T(\mu_\beta) \overset{w^*}{\to} \mu'$. Let $\nu \in M_a(S_1)$ be fixed. Then

$$\|T(\mu_\beta) * T(\nu) - T(\mu) * T(\nu)\| \to 0$$

by (2.7).

Since $M(S_1) = C_0(S_1)^*$ is a dual Banach algebra (multiplication in $M(S_2)$ is separately weak-star continuous), straightforward calculations show that $C_0(S_2)$ is a submodule of the dual Banach $M(S_2)$-module $M(S_2)^* = C_0(S_2)^{**}$. Therefore for $k \in C_0(S_2)$, $T(\nu).k \in M(S_2).C_0(S_2) \subseteq C_0(S_2)$ and hence

$$\begin{aligned}
\langle T(\mu) * T(\nu), k\rangle &= \lim\langle T(\mu_\beta) * T(\nu), k\rangle \\
&= \lim\langle T(\mu_\beta), T(\nu).k\rangle \\
&= \langle \mu', T(\nu).k\rangle \\
&= \langle \mu' * T(\nu), k\rangle.
\end{aligned}$$

Thus

$$T(\mu * \nu) = T(\mu) * T(\nu) = \mu' * T(\nu)$$

and so

$$\mu * \nu = T^{-1}(\mu') * \nu, \;\; (\nu \in M_a(S_1)). \tag{2.8}$$

Let (ν_i) be an approximate identity, then by (2.8), we have

$$\mu * \nu_i = T^{-1}(\mu') * \nu_i. \tag{2.9}$$

Also, clearly we can see that $\nu_i \to \delta_{es}$ in the weak-star topology of $M(S_1)$, by taking the weak-star limit in (2.9) we that $\mu = T^{-1}(\mu')$ and therefore $T(\mu) = \mu'$. This proves the claim. Finally, because $K^*_{\chi,\psi}$ is weak-star continuous and $T(\mu_\beta) \xrightarrow{w^*} T(\mu)$, the equality $T(\mu_\beta) = K^*_{\chi,\psi}(\mu_\beta)$ yields

$$T(\mu) = w^* - \lim T(\mu_\beta) = w^* - \lim K^*_{\chi,\psi}(\mu_\beta) = K^*_{\chi,\psi}(\mu).$$

□

Corollary 2.8. *Suppose that S_1 and S_2 are locally compact cancellative foundation semigroups. If T is a bipositive algebra isomorphism from $M(S_1)$ onto $M(S_2)$, then T is an isometry.*

Proof. By Theorem 2.6 and Theorem 2.7, there exists an isomorphism of locally compact semigroups ψ from S_1 onto S_2 such that $T = K^*_\psi$. It is readily seen that K_ψ is an isometry and therefore T is an isometric algebra isomorphism. □

Theorem 2.9. *If T is a bipositive algebra isomorphism from $M_a(S_1)$ onto $M_a(S_2)$, then there exist an isomorphism of locally compact semigroup $\psi : S_1 \to S_2$ and a continuous character $\chi : S_1 \to \mathbb{T}$ such that*

$$T(\delta_x) = \chi(x)\delta_{\phi(x)}.$$

Proof. Suppose that T is a bipositive algebra isomorphism from $M_a(S_1)$ onto $M_a(S_2)$. By Theorem 2.2, T is an isometric isomorphism from $M_a(S_1)$ onto $M_a(S_2)$ and according to Theorem 2.4, $T : \mathcal{M}(M_a(S_1)) \to \mathcal{M}(M_a(S_2))$ is an isometric isomorphism, so T is an isometric isomorphism from $M(S_1)$ onto $M(S_2)$, therefore, by Theorem 2.6, the proof is complete. □

In the following theorem, we have obtained a generalization of a well-known result of Kawada [6] and Wendel [9], for locally compact groups to a more general setting of locally compact foundation semigroups. Indeed, we prove that if T is a bipositive algebra isomorphism from $M_a(S_1)$ onto $M_a(S_2)$, then T is an isometry. So S_1 and S_2 are isomorphic.

Theorem 2.10. *If T is a bipositive algebra isomorphism from $M_a(S_1)$ onto $M_a(S_2)$, where S_1 and S_2 are locally compact cancellative foundation semigroups, then there exist a continuous character $\chi : S_1 \to \mathbb{T}$ and an isomorphism $\psi : S_1 \to S_2$ of the locally compact semigroups S_1 and S_2 such that for $\mu \in M_a(S_1)$*

$$T(\mu) = c(\chi \circ \psi^{-1})\mu \circ \psi^{-1}.$$

Proof. Suppose that T is a bipositive algebra isomorphism from $M_a(S_1)$ onto $M_a(S_2)$, we show that for any $\mu \in M(S_1)$,

$$L_\mu : M_a(S_1) \longrightarrow M_a(S_2)$$
$$\nu \mapsto T(\mu * T^{-1}(\nu))$$

where $\nu \in M_a(S_2)$, is a multiplier. According to the above theorem, there exists a measure $\tilde{T(\mu)} \in M(S_2)$ for which

$$L_\mu(\nu) = \tilde{T(\mu)} * \nu.$$

If $\mu \in M(S)^+$, then by Lemma 2.3, L_μ is a positive multiplier, so $\tilde{T(\mu)} \in M(S_2)^+$. Also, Theorem 2.4 shows that L_μ is a bipositive algebra isomorphism from $M(S_1)$ onto $M(S_2)$. In particular, it is easy to show that $\tilde{T} : \mu \longrightarrow \tilde{T}(\mu)$ is a bipositive isomorphism from $M(S_1)$ onto $M(S_2)$. Consequently by Theorem 2.6 there exist a continuous character $\chi : S_1 \to \mathbb{T}$ and an isomorphism $\psi : S_1 \to S_2$ of the locally compact semigroups S_1 and S_2. Define

$$U : M_a(S_1) \longrightarrow M_a(S_2)$$

$$U(\nu) = c(\chi \circ \psi^{-1})\nu \circ \psi^{-1}$$

where c is constant. It is readily seen that U is a bipositive algebra isomorphism. Now we prove that

$$TU^{-1}r_y UT^{-1} = r_y$$

which $r_y f(x) = f(xy)$. Since

$$\begin{aligned} c^{-1}\bar{\chi}(c\chi \circ \psi^{-1}T^{-1}(\nu) \circ \psi^{-1}) \circ \psi)(z) =& c^{-1}\bar{\chi}(z)(c\chi \circ \psi^{-1}T^{-1}(\nu) \circ \psi^{-1})(\psi(z)) \\ =& (c^{-1}\bar{\chi}(z))(c\chi(z))T^{-1}(\nu)(z) \\ =& T^{-1}(\nu)(z) \end{aligned}$$

we have

$$\begin{aligned} TU^{-1}r_y UT^{-1}(\nu)(x) =& TU^{-1}r_y(c\chi o\psi^{-1}T^{-1}(\nu) \circ \psi^{-1})(x) \\ =& TU^{-1}(c\chi \circ \psi^{-1}T^{-1}(\nu) \circ \psi^{-1})(xy) \\ =& T(c^{-1}\bar{\chi}(c\chi \circ \psi^{-1}T^{-1}(\nu) \circ \psi^{-1}) \circ \psi)(xy) \\ =& T(T^{-1}\nu)(xy) = \nu(xy) = r_y\nu(x). \end{aligned}$$

So TU^{-1} is a left multiplier. Note that the left multiplier TU^{-1} is a bipositive invertible left multiplier, the measure μ is also positive because TU^{-1} is bipositive thus μ^{-1} is positive. On the other hand μ is invertible because TU^{-1} is an invertible left multiplier. So TU^{-1} is the identity operator on $M_a(S_2)$, and $T = U$. □

Corollary 2.11. *If T is a bipositive algebra isomorphism from $M_a(S_1)$ onto $M_a(S_2)$, where S_1 and S_2 are locally compact cancellative foundation semigroups, then T is an isometry.*

Proof. By Theorem 2.6 and Theorem 2.10, there exists an isomorphism ψ from S_1 onto S_2 for which for any $\nu \in M_a(S_1)$ and constant c, we have $T = c\nu \circ \psi^{-1}$. Now it is easy to see that T is an isometric algebra isomorphism. □

3. Bipositive isomorphisms on second dual of semigroup algebras

Definition 3.1. *Suppose that A is a Banach algebra, then the dual space A^* can be a right Banach A-bimodule with the canonical operation*

$$\langle f \diamond a, b\rangle = \langle f.a, b\rangle = \langle f, ab\rangle,$$

where $f \in A^*$ and a, $b \in A$. Let Y be a norm closed A-submodule of A^*, then we may define $n \diamond f \in A^*$ by

$$\langle n \diamond f, a\rangle = \langle n, f \diamond a\rangle,$$

for $n \in Y^*$ and $f \in Y$. If $n \diamond f \in Y$, then Y is called a left introverted subspace of A^*, by defining $m \diamond n \in Y^*$ through

$$\langle m \diamond n, f\rangle = \langle m, n \diamond f\rangle, \qquad (m, n \in Y^*,\ f \in Y).$$

Theorem 3.2. *If the Banach algebras $LUC(S_1)^*$ and $LUC(S_2)^*$ are bipositively algebraically isomorphic, then there exist topological group isomorphism ψ from S_1 onto S_2, a continuous homomorphism $\chi : S_1 \longrightarrow (0, \infty)$, and positive constants m and M such that*

$$m \leq \chi(x) \leq M \qquad (x \in S).$$

Proof. Let $T : LUC(S_1)^* \longrightarrow LUC(S_2)^*$ be a bipositive algebra isomorphism. Because both T and T^{-1} are order-preserving and positive operators, so they are bounded, and therefore there exists the set of extreme points of unit ball of the measure algebra $M(S_1)$ such that for every $x \in S_1$ there exist $\psi(x) \in S_1$, $\chi(x) \in \mathbb{T}$, that is

$$T(\delta_x) = \chi(x)\delta_{\psi(x)}.$$

To prove the continuity of χ and ψ, we assume that a net (x_α) in S_1 converges to x in S_1, and we can also assume that x_α are contained in a compact neighborhood U of x. Then for every μ in $M_a(S_1)$, we get the result

$$\delta_{x_\alpha} * \mu \longrightarrow \delta_x * \mu.$$

Since T is a bounded algebra isomorphism,

$$T(\delta_{x_\alpha}) \diamond T(\mu) \longrightarrow T(\delta_x) \diamond T(\mu)$$

where $(T(\delta_{x_\alpha}))$ is a bounded net in $M(S_2)$. On the other hand, since

$$\|T(\delta_{x_\alpha})\| \leq \|T\|,$$

we can assume that there is a subnet $(T(\delta_{x_{\alpha(i)}}))$ and an element m in $LUC(S_2)^*$ such that $T(\delta_{x_{\alpha(i)}}) \longrightarrow m$ in the weak-star topology of $LUC(S_2)^*$. Since T is an isomorphism, we get that both χ and ψ are multiplicative, thus

$$T(\delta_{x_{\alpha(i)}}) \diamond T(\mu) \xrightarrow{w^*} m \diamond T(\mu).$$

This implies that

$$\delta_x \diamond \mu = T^{-1}(m) \diamond \mu$$

for every μ in $M_a(S_1)$. We know that $LUC(S_1) = M_a(S_1) \diamond LUC(S_1)$, see [7], so $T(\delta_x) = m$.

Using the same argument as above, it follows that every subnet of $(T(\delta_x))$ has a convergent subnet to $T(\delta_x)$, so we conclude that

$$\chi(x_\alpha)\delta_{\psi(x_\alpha)} = T(\delta_{x_\alpha}) \xrightarrow{w^*} T(\delta_x) = \chi(x)\delta_{\psi(x)}.$$

By using Theorem 2.10 and its proof, we can conclude the continuity of χ and ψ and also by considering T^{-1} we can show that ψ is surjective with a continuous inverse. □

It is well-known from [7], that an element E in $M_a(S)^{**}$ is called a mixed identity with norm one if

$$\nu \odot E = E \odot \nu = \nu \qquad (\nu \in M_a(S)).$$

On the other hand

$$LUC(S) = M_a(S) \circ L^\infty(S, M_a(S)).$$

In this case, there exists an isometric isomorphism π_E such that

$$\pi_E : EL^\infty(S, M_a(S))^* \longrightarrow LUC(S)^*.$$

Now, suppose that there exists an isometric isomorphism T from $EL^\infty(S_1, M_a(S_1))^*$ onto $L^\infty(S_2, M_a(S_2))^*$, and $E \in L^\infty(S_1, M_a(S_1))^*$ is a right identity with norm one, then $T(E) \in L^\infty(S_2, M_a(S_2))^*$ is a right identity with $\|T(E)\| = 1$.

In [4], H. Farhadi showed that for two locally compact groups G_1 and G_2, if T defined by $T : L^1(G_1)^{**} \longrightarrow L^1(G_2)^{**}$ is a bipositive algebra isomorphism, then T is an isometry and the groups G_1 and G_2 are topologically isomorphic. In the following theorem, we give a generalization of this theorem on the second dual of semigroup algebras.

Theorem 3.3. *Let $T : M_a(S_1)^{**} \longrightarrow M_a(S_2)^{**}$ be a bipositive algebra isomorphism. Then the semigroups S_1 and S_2 are topologically isomorphic.*

Proof. To prove it, first assume that there is a net (μ_α) in $M_a(S_1)$ such that $\mu_\alpha \longrightarrow \mu$ in $M_a(S_1)$ with $\|\mu_\alpha\| = \|\mu\| = 1$. Thus

$$T(\mu_\alpha) \longrightarrow T(\mu).$$

Let E denote a weak-star cluster point of the canonical image of the bounded approximate identity (μ_α) of $M_a(S_1)$ in $M_a(S_1)^{**}$ where $\mu_\alpha \geq 0$. Then E is a positive right identity for $M_a(S_1)^{**}$, now we show that $T(E)$ is a positive right identity of $M_a(S_2)^{**}$, since $\|\mu_\alpha\| = 1$, as a result $\|E\| \leq 1$, so we get

$$\|E\| = \|E \diamond E\| \leq \|E\|\|E\|$$

and

$$\|E\| \geq 1.$$

Next, we prove that there is a bipositive algebra isomorphism from $LUC(S_1)^{**}$ onto $LUC(S_2)^{**}$, for this we define the following mappings

$$\kappa_E : EM_a(S_1)^{**} \longrightarrow LUC(S_1)^*$$
$$E_n \longrightarrow n\big|_{LUC(S_1)^*}$$

and

$$\kappa_{T(E)} : T(E)M_a(S_2)^{**} \longrightarrow LUC(S_2)^*$$
$$T(E)_m \longrightarrow m\big|_{LUC(S_2)^*}.$$

According to the above bipositive algebra isomorphisms, $\kappa_E^{-1} \circ T \circ \kappa_{T(E)}$ is a bipositive algebra isomorphism from $LUC(S_1)^*$ onto $LUC(S_2)^*$. Therefore, according to Theorem 3.2, there exist the numbers m and M and also an isomorphism ψ from S_1 onto S_2, a semigroup isomorphism $\chi : S_1 \longrightarrow (0, \infty)$, such that

$$m \leq \chi(x)\delta_{\psi(x)} \leq M$$

for $x \in S_1$. □

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