Measure Algebras and Applications, Year. 2025, Vol. 3, No. 1, pp. 80-92.

Measure Algebras and Applications

On weighted Banach spaces of vector-valued holomorphic functions

Esmaeil Feizi[1]✉, **Zahra Sadat Hosseini**[1]

1. Department of Mathematics, Bu-Ali Sina University, Hamedan, Iran.

Email: efeizi@basu.ac.ir (Corresponding Author), z.hoseini@basu.ac.ir

Article Info

Article history:
Received: 2 February 2025
Received in revised form:
24 April 2025
Accepted: 6 May 2025
Published Online:
25 June 2025

Keywords:
Holomorphic function,
Composition operator,
Differentiation operator,
Vector-valued function

2020 Mathematics Subject Classification:
46E15, 47B38, 32A38

ABSTRACT

Let X be a complex Banach space, $\mathbb{D}$ the unit disc, and v a weight function. We investigate the weighted spaces of vector-valued holomorphic functions $H_v(\mathbb{D}, X)$. In particular, we study the boundedness and compactness of differentiation and composition operators on these spaces. Additionally, we examine the relationships among the algebraic structure of $H_v(\mathbb{D}, \mathcal{A})$, the Banach algebra $\mathcal{A}$, and the weight function v.

Cite this article: Feizi, E., & Sadat Hosseini, Z. (2025). On weighted Banach spaces of vector-valued holomorphic functions. *Measure Algebras and Applications*, 3(1), 80–92. http://doi.org/10.22091/maa.2026.14814.1043

Publisher: University of Qom
Doi:http://doi.org/10.22091/maa.2026.14814.1043

1. Preliminaries

Let X be a complex Banach space and U an open subset of the complex plane $\mathbb{C}$. A function $f : U \to X$ is said to be *holomorphic* if

$$f'(z) := \lim_{\substack{h\to 0\\ h\in\mathbb{C}\setminus\{0\}}} \frac{f(z+h)-f(z)}{h} \tag{1.1}$$

exists for all $z \in U$. The space of all X-valued holomorphic functions on U is denoted by $H(U, X)$. This space is a complete, locally convex space with respect to the compact-open topology [20]. We will write $H(U)$ as shorthand for $H(U, \mathbb{C})$.

A strictly positive function $\upsilon : U \to (0, \infty)$ is called a *weight function.* The subspace $H_\upsilon(U, X)$ is the normed space consisting of all X-valued holomorphic functions $f \in H(U, X)$ that are bounded with respect to the extended weighted sup-norm

$$\|f\|_\upsilon = \sup_{z\in U} \|f(z)\|\, \upsilon(z),$$

where $\|\cdot\|$ denotes the norm in the Banach space X. We write $H_\upsilon(U)$ when $X = \mathbb{C}$.

Some classical results from complex analysis do not extend directly to the vector-valued setting—for example, Montel's theorem fails in this context. This presents a significant challenge when attempting to generalize results from the scalar-valued to the vector-valued framework. Hence, transferring results to the vector-valued setting is not always straightforward or trivial.

The purpose of this paper is to extend several known results for complex-valued holomorphic functions to the vector-valued case. Vector-valued holomorphic functions are important tools in the theory of functional calculus and play a vital role in the theory of one-parameter semigroups [3, 23]. Considerable research has been conducted on the structure and properties of vector-valued holomorphic function spaces. For more information, see [3, 6, 8, 17, 19].

A particularly interesting direction of generalization involves classical operators. The composition operator $C_\varphi : f \mapsto f \circ \varphi$, where $\varphi : \mathbb{D} \to \mathbb{D}$ is holomorphic, has been studied by Ryff [22] and Nordgren [21]. The differentiation operator $D : f \mapsto f'$ has been investigated by Harutyunyan and Lusky [18]. Necessary and sufficient conditions for the boundedness and compactness of these operators have also been discussed (see, e.g., [7, 9, 10]).

The paper is organized as follows. In Section 2, we begin with a discussion of the space $H_\upsilon(U, X)$, where U is an open subset of $\mathbb{C}$ and υ is a weight function. We then introduce the subspace $H_\upsilon^{2\pi}(\mathbb{G}, X)$, consisting of all 2π-periodic vector-valued holomorphic functions on the upper half-plane $\mathbb{G}$, and prove that it is isomorphic to $H_{\hat\upsilon}(\mathbb{D}, X)$ for a particular weight function $\hat\upsilon$.

In Section 3, we study two classical operators—differentiation and composition—on the space $H_\upsilon(\mathbb{D}, X)$. Specifically, we determine necessary conditions for the compactness of these operators, necessary and sufficient conditions for the boundedness of the composition operator, and necessary conditions for the boundedness of the differentiation operator.

In Section 4, we focus on the algebraic structure of the space $H_\upsilon(\mathbb{D}, \mathcal{A})$, where $\mathcal{A}$ is a Banach algebra. We analyze the algebraic properties of $H_\upsilon(\mathbb{D}, \mathcal{A})$ in terms of the structure of $\mathcal{A}$ and the weight function υ. For example, we show that $H_\upsilon(\mathbb{D}, \mathcal{A})$ is a unital Banach algebra if and only if $\mathcal{A}$ is unital and υ is a bounded weight function.

2. Isomorphism of $H_{\hat\upsilon}(\mathbb{D}, X)$ and $H_\upsilon^{2\pi}(\mathbb{G}, X)$

Let U be an open subset of the complex plane $\mathbb{C}$ and X a complex Banach space. For a weight function υ on U, the weighted normed space of vector-valued holomorphic functions is defined by

$$H_\upsilon(U,X) := \left\{ f \in H(U,X) : \|f\|_\upsilon = \sup_{z\in U} \|f(z)\|\, \upsilon(z) < \infty \right\}.$$

The following general result is fundamental in the study of this class of vector-valued holomorphic functions.

Theorem 2.1. *Suppose that U is an open subset of $\mathbb{C}$, υ is a weight function, and X is a complex Banach space. Then $H_\upsilon(U,X)$ is a Banach space.*

Throughout this paper, we denote by $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the unit disc and by $\mathbb{G} = \{\omega \in \mathbb{C} : \mathrm{Im}(\omega) > 0\}$ the upper half-plane.

A weight υ on $\mathbb{D}$ is called *radial* if

$$\upsilon(\lambda z) = \upsilon(z), \quad \text{for all } \lambda \in \mathbb{C} \text{ with } |\lambda| = 1.$$

A continuous weight υ on the upper half-plane $\mathbb{G}$ is called *standard* if $\lim_{r\to 0} \upsilon(ir) = 0$ and there exists a constant $c > 0$ such that

$$\upsilon(z) \le c\,\upsilon(w) \quad \text{whenever } 0 < \mathrm{Im}\, z \le \mathrm{Im}\, w.$$

Example 2.2. *We present two standard examples on the unit disc and the upper half-plane.*

(i) $\upsilon_1(z) = (1-|z|^2)^p$ *is a standard, continuous, radial weight on* $\mathbb{D}$ *for all* $0 < p < \infty$.

(ii) $\upsilon_2(z) = \exp\left(-\frac{\alpha}{Im\, z}\right)$ *is a standard weight on* $\mathbb{G}$ *for all* $\alpha > 0$.

Now suppose that υ is a continuous radial weight on $\mathbb{D}$. The corresponding *associated weight* $\tilde{\upsilon}$ is defined by

$$\tilde{\upsilon}(z) = \left(\sup\{|f(z)| : \|f\|_\upsilon \le 1\}\right)^{-1}, \quad (z \in \mathbb{D}).$$

Note that the term *radial* is used only for weights on the unit disc. In this work, υ denotes a standard weight on $\mathbb{G}$ depending only on the imaginary part of the variable, whereas the associated weight $\hat{\upsilon}$ is radial on the unit disc $\mathbb{D}$.

It is easy to verify that

$$\frac{1}{\tilde{\upsilon}(z)} = \|\delta_z\|_{H_\upsilon(\mathbb{D})^*},$$

where δ_z denotes the point evaluation at $z \in \mathbb{D}$. Bierstedt et al. [5] proved that associated weights are continuous, satisfy $\upsilon \le \tilde{\upsilon}$, and that for each $z \in \mathbb{D}$, there exists $f_z \in H_\upsilon(\mathbb{D})$ such that $\|f_z\|_\upsilon \le 1$ and $|f_z(z)| = \frac{1}{\tilde{\upsilon}(z)}$.

Applying the Hahn-Banach theorem, we conclude that if X is a complex Banach space, then for every $f \in H_\upsilon(\mathbb{D}, X)$ we have

$$\sup_{z\in\mathbb{D}} \|f(z)\|\, \tilde{\upsilon}(z) = \sup_{z\in\mathbb{D}} \|f(z)\|\, \upsilon(z).$$

This implies that $H_{\tilde{\upsilon}}(\mathbb{D}, X)$ is isometrically isomorphic to $H_\upsilon(\mathbb{D}, X)$ [11].

A subspace of $H_\upsilon(\mathbb{G}, X)$, namely the weighted space of vector-valued 2π-periodic holomorphic functions from $\mathbb{G}$ to X, is defined by

$$H_\upsilon^{2\pi}(\mathbb{G}, X) := \{f \in H_\upsilon(\mathbb{G}, X) : f(z) = f(z+2\pi) \text{ for all } z \in \mathbb{G}\}.$$

Suppose υ is a standard weight such that $H_\upsilon^{2\pi}(\mathbb{G}, X) \ne \{0\}$. Ardalani and Lusky, in [2], defined b_υ as the smallest integer satisfying

$$\sup_{z\in\mathbb{G}} e^{-b_\upsilon \mathrm{Im}(z)}\, \upsilon(z) < \infty. \tag{2.1}$$

It is well known that every holomorphic function admits a unique power series expansion, and if a complex function has an isolated singularity, then the Taylor series may be replaced by a Laurent series. These facts extend naturally to the vector-valued case. The following two results, originally stated for scalar-valued functions in [2], hold in the vector-valued setting as well.

Proposition 2.3. *For each $f \in H_\upsilon^{2\pi}(\mathbb{G}, X)$, there exist elements $\lambda_k \in X$ such that for all $\omega \in \mathbb{G}$,*

$$f(\omega) = \sum_{k=b_\upsilon}^{\infty} \lambda_k e^{ik\omega},$$

where the series converges uniformly on compact subsets of $\mathbb{G}$.

Proof. First, suppose $z \in \mathbb{D} \setminus \{0\}$ and define the conformal map $\tau(z) := -i \log z$. If $f \in H_\upsilon^{2\pi}(\mathbb{G}, X)$, then the composition $(f \circ \tau)(z) = f(-i \log z)$ is holomorphic. Using the Laurent series expansion in the vector-valued case for $z = e^{i\omega}$, we obtain:

$$f(\omega) = f(-i \log z) = \sum_{n=-\infty}^{\infty} \lambda_n e^{in\omega}, \tag{2.2}$$

where $\{\lambda_n\}_{n\in\mathbb{Z}}$ is a sequence in X.

Using this representation and the orthogonality of exponentials, we compute:

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} f(\omega + x)e^{-ikx}\, dx &= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \lambda_n e^{in\omega} \int_0^{2\pi} e^{i(n-k)x}\, dx \\ &= \lambda_k e^{ik\omega}.\end{aligned}$$

Taking the norm on both sides and multiplying by $\upsilon(\omega)$ yields:

$$\begin{aligned}\|\lambda_k e^{ik\omega}\|\, \upsilon(\omega) &= \left\| \frac{1}{2\pi}\int_0^{2\pi} f(\omega + x)e^{-ikx}\, dx \right\| \upsilon(\omega) \\ &\le \frac{1}{2\pi}\int_0^{2\pi} \|f(\omega + x)\|\, dx\, \upsilon(\omega) \\ &\le \sup_{0\le x\le 2\pi} \|f(\omega + x)\|\, \upsilon(\omega) \\ &\le c \sup_{0\le x\le 2\pi} \|f(\omega + x)\|\, \upsilon(\omega + x), \qquad (2.3)\end{aligned}$$

where the last inequality holds for some constant $c > 0$ due to the standardness of the weight υ. Thus, for all $\omega \in \mathbb{G}$, we have

$$\begin{aligned}\|\lambda_k\|\, e^{-k\,\mathrm{Im}(\omega)}\, \upsilon(\omega) &= \|\lambda_k e^{ik\omega}\|\, \upsilon(\omega) \\ &\le c \sup_{0\le x\le 2\pi} \|f(\omega + x)\|\, \upsilon(\omega + x).\end{aligned}$$

Now, by the definition in (2.1), it follows that $b_\upsilon \le k$. Hence, from the series in (2.2), the proof is complete. □

We conclude this section by extending a classical theorem concerning the isomorphism between the spaces $H_\upsilon^{2\pi}(\mathbb{G}, X)$ and $H_{\hat{\upsilon}}(\mathbb{D}, X)$.

Theorem 2.4. *Suppose X is a complex Banach space and υ is a weight function on $\mathbb{G}$. For every $z \in \mathbb{D}$, define*

$$\hat{\upsilon}(z) = \begin{cases} |z|^{b_\upsilon}\, \upsilon(-i \log |z|), & \text{if } z \ne 0, \\ 0, & \text{if } z = 0. \end{cases}$$

Then $\hat{\upsilon}$ is a radial weight on $\mathbb{D}$ such that

$$\lim_{|z|\to 1} \hat{\upsilon}(z) = 0.$$

Moreover, the two spaces $H_v^{2\pi}(\mathbb{G}, X)$ and $H_{\hat{v}}(\mathbb{D}, X)$ are isomorphic.

Proof. Since v is a standard weight, it is equivalent to $v_1(\omega) := v(i, \operatorname{Im}\omega)$ (see [2]). Let $f \in H_v^{2\pi}(\mathbb{G}, X)$. By Proposition 2.3, we can write

$$f(\omega) = \sum_{k=b_v}^{\infty} \lambda_k e^{ik\omega}, \qquad \lambda_k \in X.$$

Setting $z = e^{i\omega}$, we have $z \in \mathbb{D}$ and

$$f(-i \log z) = \sum_{k=b_v}^{\infty} \lambda_k z^k.$$

Hence,

$$z^{-b_v} f(-i \log z) = \sum_{k=b_v}^{\infty} \lambda_k z^{k-b_v} = \sum_{n=0}^{\infty} \lambda_{n+b_v} z^n.$$

This shows that $z = 0$ is a removable singularity for the operator defined below, and therefore the resulting function is holomorphic on $\mathbb{D}$.

Define the operator

$$S : H_v^{2\pi}(\mathbb{G}, X) \to H_{\hat{v}}(\mathbb{D}, X), \qquad (Sf)(z) := z^{-b_v} f(-i \log z), \quad z \in \mathbb{D}.$$

Using the definitions of $\hat{v}$ and S, we compute

$$\begin{aligned} \|(Sf)(z)\| \, \hat{v}(z) &= \|f(-i \log z)\| \, v(-i \log |z|) \\ &= \left\| \sum_{k=b_v}^{\infty} \lambda_k z^k \right\| v(-i \log |z|). \end{aligned}$$

Taking the supremum over $\mathbb{D}$, we obtain

$$\|Sf\|_{\hat{v}} = \|f\|_{v_1}.$$

Thus, S is injective and continuous. Since v and v_1 are equivalent, the norms $\|\cdot\|_v$ and $\|\cdot\|_{v_1}$ are equivalent as well.

Now let $g \in H_{\hat{v}}(\mathbb{D}, X)$ and define $f(\omega) := e^{ib_v\omega} g(e^{i\omega})$, $\omega \in \mathbb{G}$. It is easy to verify that $f \in H_v^{2\pi}(\mathbb{G}, X)$ and $(Sf)(z) = g(z)$ for all $z \in \mathbb{D}$. Hence, S is surjective.

Therefore, S is an isomorphism from $H_v^{2\pi}(\mathbb{G}, X)$ onto $H_{\hat{v}}(\mathbb{D}, X)$. □

Note that the weight $\hat{v}$ in the above theorem is not necessarily continuous. However, there exists a continuous radial weight u on $\mathbb{D}$ defined by

$$u(z) = \begin{cases} \sup\limits_{\frac{1}{2} \le t < 1} v(-i \log t) & \text{if } |z| \le \frac{1}{2}, \\ \sup\limits_{|z| \le t < 1} v(-i \log t) & \text{if } \frac{1}{2} < |z| < 1, \end{cases}$$

such that $\hat{v}|_{[1/2,1)}$ and $u|_{[1/2,1)}$ are equivalent. Moreover, the norms $\|\cdot\|_{\hat{v}}$ and $\|\cdot\|_u$ are also equivalent; see [2, Lemma 2.3].

3. Composition and Differentiation operators on $H_\upsilon(\mathbb{D},X)$

Let X be a complex Banach space, and let $\varphi : \mathbb{D} \to \mathbb{D}$ be a holomorphic mapping. The composition operator associated with φ is defined by

$$C_\varphi : H(\mathbb{D}) \to H(\mathbb{D}), \quad f \mapsto C_\varphi(f) = f \circ \varphi.$$

We also define the differentiation operator as

$$D : H(\mathbb{D}) \to H(\mathbb{D}), \quad f \mapsto D(f) = f'.$$

Let $\widetilde{C_\varphi}$ and $\widetilde{D}$ denote the corresponding operators on the vector-valued space $H(\mathbb{D},X)$.

Wolf, in [24], established necessary and sufficient conditions for the compactness of the composition-differentiation operator DC_φ on $H_\upsilon(\mathbb{D})$. In the special case when $\varphi = \mathrm{id}_\mathbb{D}$, Theorem 2 of [24] shows that the compactness of the differentiation operator

$$D : H_\upsilon(\mathbb{D}) \to H_\omega(\mathbb{D})$$

depends critically on the choice of weights υ and ω.

For example, by that theorem, when $\upsilon = \omega$, the operator D is not compact. However, for

$$\upsilon(z) = 1 - |z|^2 \quad \text{and} \quad \omega(z) = (1 - |z|^2)^3,$$

the operator D is compact.

Before we present the main results, we recall the following useful properties. Let $g \in H_\upsilon(\mathbb{D},X)$, $f \in H_\upsilon(\mathbb{D})$, and $z \in \mathbb{D}$.

(i) If υ is a bounded weight, then $H_\upsilon(\mathbb{D})$ contains all constant functions.

(ii) For any nonzero $x \in X$, the map

$$J_x : H_\upsilon(\mathbb{D}) \to H_\upsilon(\mathbb{D},X), \quad f \mapsto f \otimes x, \quad (f \otimes x)(z) = f(z)x,$$

is a bounded linear injection with $\|J_x\| \le \|x\|$.

(iii) For any $x^* \in X^*$, the map

$$Q_{x^*} : H_\upsilon(\mathbb{D},X) \to H_\upsilon(\mathbb{D}), \quad g \mapsto x^* \circ g, \quad (x^* \circ g)(z) = x^*(g(z)),$$

is a bounded surjection with $\|Q_{x^*}\| \le \|x^*\|$.

(iv) The point evaluation map

$$\delta_z : H_\upsilon(\mathbb{D},X) \to X, \quad \delta_z(g) = g(z),$$

is bounded with $\|\delta_z\| \le \frac{1}{\upsilon(z)}$.

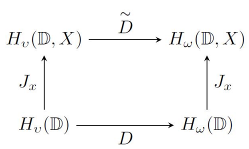
For further details, see [19]. It is also worth noting that

$$Q_{x^*} \circ J_x = \mathrm{id}_{H_\upsilon(\mathbb{D})}, \quad \text{whenever } \langle x, x^* \rangle = 1.$$

Using these tools, a necessary condition for the compactness of the composition operator $\widetilde{C_\varphi}$ on the Banach space $H_\upsilon(\mathbb{D},X)$ was established. Under the same assumptions, we now present the following results.

Lemma 3.1. *Suppose that X is a complex Banach space, and υ and ω are continuous weights on $\mathbb{D}$. The following assertions hold:*

(a) For all $x \in X$, the following diagram commutes.



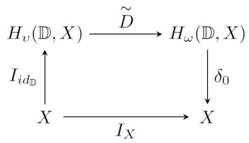
(b) Suppose that $\mathrm{id}_{\mathbb{D}} \in H_\upsilon(\mathbb{D})$ *and*

$$\widetilde{D} : H_\upsilon(\mathbb{D}, X) \to H_\omega(\mathbb{D}, X)$$

is a well-defined operator. Define the linear operator

$$I_{\mathrm{id}_{\mathbb{D}}} : X \to H_\upsilon(\mathbb{D}, X), \quad I_{\mathrm{id}_{\mathbb{D}}}(x)(z) = (\mathrm{id}_{\mathbb{D}} \otimes x)(z) = xz,$$

for all $z \in \mathbb{D}$*. Then, the following diagram commutes.*



Proof.

(a) For all $x \in X$ and $f \in H_\upsilon(\mathbb{D})$, we compute

$$\widetilde{D} \circ J_x(f) = \widetilde{D}(f \otimes x).$$

By definition, we have

$$\begin{aligned}
\widetilde{D}(f \otimes x)(z) &= \lim_{w \to z} \frac{(f \otimes x)(w) - (f \otimes x)(z)}{w - z} \\
&= \lim_{w \to z} \frac{f(w)x - f(z)x}{w - z} \\
&= \left(\lim_{w \to z} \frac{f(w) - f(z)}{w - z} \right) x = D(f)(z)x = (D(f) \otimes x)(z) \\
&= (J_x \circ D)(f)(z).
\end{aligned}$$

Hence, $\widetilde{D} \circ J_x = J_x \circ D$, and the diagram commutes.

(b) Let $x \in X$ be arbitrary. Then

$$\begin{aligned}
\|I_{\mathrm{id}_{\mathbb{D}}}(x)\|_\upsilon &= \sup_{z \in \mathbb{D}} \|\mathrm{id}_{\mathbb{D}}(z)x\|\upsilon(z) \\
&= \|x\| \cdot \|\mathrm{id}_{\mathbb{D}}\|_\upsilon < \infty,
\end{aligned}$$

so $I_{\mathrm{id}_{\mathbb{D}}}$ is a bounded, injective, linear operator.

According to the definitions in the diagram of part (b), for all $x \in X$, we have:

$$\delta_0 \circ \widetilde{D} \circ I_{\mathrm{id}_{\mathbb{D}}}(x) = \delta_0 \circ \widetilde{D}(\mathrm{id}_{\mathbb{D}} \otimes x) = \delta_0(x) = x = I(x).$$

This shows that $\delta_0 \circ \widetilde{D} \circ I_{\mathrm{id}_{\mathbb{D}}} = I$, and thus the diagram commutes. □

The following result is an immediate consequence of the commutativity of the diagrams in Lemma 3.1.

Corollary 3.2. *With the notation in Lemma 3.1, we have:*

(i) If $\widetilde{D}$ *is bounded, then* D *is also bounded.*

(ii) If $\widetilde{D} : H_\upsilon(\mathbb{D}, X) \to H_\omega(\mathbb{D}, X)$ *is compact, then* D *is compact and* X *is finite-dimensional.*

(iii) If $\widetilde{D} : H_\upsilon(\mathbb{D}, X) \to H_\omega(\mathbb{D}, X)$ *is weakly compact, then* D *is weakly compact and* X *is reflexive.*

Proof.

(i) Suppose $\widetilde{D}$ is bounded. Since J_x is injective, it has a left inverse

$$Q_{x^*} : H_\omega(\mathbb{D}, X) \to H_\omega(\mathbb{D}), \quad Q_{x^*}(f) = x^* \circ f,$$

which is bounded by the inverse mapping theorem. Then $Q_{x^*} \circ \widetilde{D} \circ J_x$ is bounded, so D is bounded.

(ii) Suppose $\widetilde{D}$ is compact, and let L_x^{-1} be a left inverse of J_x for some $x \in X$. By the commutativity of the diagram in Lemma 3.1(a), we have

$$L_x^{-1} \circ \widetilde{D} \circ J_x = D,$$

and by Proposition 3.5 in [15], D is compact.

From part (b) of Lemma 3.1, and a similar argument, it follows that the operator I_X is compact, hence X is finite-dimensional.

(iii) Using the same reasoning and Proposition 5.2 in [15], the operators D and I_X are weakly compact. Then, by Proposition 5.5 in [15], X is a reflexive Banach space. □

Before proceeding further, it is worthwhile to extend Theorem 2.1 from [14, Chapter VII]. Recall that for a Banach space X, the space of all X-valued continuous functions on an open subset $U \subseteq \mathbb{D}$, equipped with the compact-open topology, is denoted by $C(U, X)$.

Theorem 3.3. *Let $\{f_n\}$ be a sequence in $H(U, X)$, and suppose $f \in C(U, X)$ such that $f_n \to f$ with respect to the compact-open topology τ_{co}. Then f is analytic and, for each integer $k \geq 1$, $f_n^{(k)} \to f^{(k)}$ with the same topology.*

Proof. Since Morera's Theorem and Cauchy's Integral Formula hold for vector-valued holomorphic functions (see [1, 1.5]), the result follows directly from [14, Theorem 2.1]. □

Now, we establish some necessary conditions for the boundedness of the differentiation operator $\widetilde{D}$ in the vector-valued setting.

Theorem 3.4. *Let v be a radial, continuous, non-increasing weight on $\mathbb{D}$ such that*

$$\lim_{|z| \to 1^-} v(z) = 0,$$

and define $v_1(z) = (1 - |z|)v(z)$ for all $z \in \mathbb{D}$. Then the following statements are equivalent:

(a) For every $f \in H_v(\mathbb{D}, X)$, we have $\widetilde{D}f \in H_{v_1}(\mathbb{D}, X)$,

(b) $\widetilde{D}$ is a bounded operator from $H_v(\mathbb{D}, X)$ into $H_{v_1}(\mathbb{D}, X)$.

Moreover, each of the above implies:

(c)

$$\sup_{n \in \mathbb{N}} \frac{v(1 - 2^{-n})}{v(1 - 2^{-n-1})} < \infty.$$

Proof. **(a) ⇒ (b)**: Let $\{f_n\} \subseteq H_v(\mathbb{D}, X)$ be a sequence such that $(f_n, f_n') \to (f, g)$. To prove that $\widetilde{D}$ is continuous, it suffices to show $g = f'$. Since $f_n \xrightarrow{\tau_v} f$ and $\tau_{co} \subseteq \tau_v$, we also have $f_n \xrightarrow{\tau_{co}} f$. By Theorem 3.3, $f_n' \xrightarrow{\tau_{co}} f'$, and since $f_n' \xrightarrow{\tau_v} g$, it follows that $f_n' \xrightarrow{\tau_{co}} g$ as well. Thus, $f' = g$ because $H_{v_1}(\mathbb{D}, X)$ is Hausdorff. Hence, by the Closed Graph Theorem, $\widetilde{D}$ is bounded.

(b) ⇒ (a) is immediate.

(b) ⇒ (c): Assume that $\widetilde{D}$ is bounded. Let $x \in X$, $x^* \in X^*$ such that $\langle x, x^* \rangle = 1$. From the commutative diagram in Lemma 3.1(a), we have

$$\widetilde{D} \circ J_x = J_x \circ D.$$

Thus,

$$\begin{aligned} Q_{x^*} \circ (\widetilde{D} \circ J_x) &= Q_{x^*} \circ (J_x \circ D) \\ &= (Q_{x^*} \circ J_x) \circ D \\ &= D. \end{aligned}$$

Therefore, D is a bounded operator from $H_v(\mathbb{D})$ to $H_{v_1}(\mathbb{D})$, and by [18, Theorem 3.2], the condition in (c) follows. □

In the rest of this section, we explore the similarities and connections between the two operators C_φ and $\widetilde{C}_\varphi$. Laitila and Tylli showed in [19] that $\widetilde{C}_\varphi$ is never compact on $H_v(\mathbb{D}, X)$ when X is an infinite-dimensional complex Banach space. Therefore, we investigate the distinction between compactness and weak compactness of C_φ and $\widetilde{C}_\varphi$.

Theorem 3.5. *Let φ be an analytic self-map of $\mathbb{D}$. Let v be a radial, continuous weight, decreasing on $[0,1]$, with*

$$\lim_{r\to 1^-} v(r) = 0,$$

and let ω be a weight on $\mathbb{D}$ vanishing at the boundary. Then:

(a) The operator $\widetilde{C}_\varphi$ is bounded from $H_v(\mathbb{D}, X)$ to $H_\omega(\mathbb{D}, X)$ if and only if

$$\sup_{z\in\mathbb{D}} \frac{\omega(z)}{v(\varphi(z))} < \infty. \tag{3.1}$$

(b) Let v and ω be continuous, non-increasing weights on $\mathbb{D}$, both tending to zero at the boundary, i.e.,

$$\lim_{|z|\to 1^-} v(|z|) = \lim_{|z|\to 1^-} \omega(|z|) = 0.$$

Consider the following conditions:

(i) $\widetilde{C}_\varphi : H_v(\mathbb{D}, X) \to H_\omega(\mathbb{D}, X)$ is compact.

(ii) $\widetilde{C}_\varphi : H_v(\mathbb{D}, X) \to H_\omega(\mathbb{D}, X)$ is weakly compact.

(iii)

$$\lim_{n\to\infty} \frac{\|e^{in\varphi(z)}\|_\omega}{\|e^{inz}\|_v} = 0.$$

Then (i) ⇒ (ii) ⇒ (iii). However, the equivalence between these conditions—as in the scalar-valued case—does not hold in general.

Proof. (a) Suppose $\widetilde{C}_\varphi$ is bounded from $H_v(\mathbb{D}, X)$ into $H_\omega(\mathbb{D}, X)$. Then, by [19, Corollary 2], the scalar operator C_φ is also bounded. Therefore, [7, Theorem 4] implies that relation (3.1) must hold.

Conversely, let $f \in H_v(\mathbb{D}, X)$. Then:

$$\begin{aligned} \|\widetilde{C}_\varphi(f)\|_\omega &= \sup_{z\in\mathbb{D}} \|f \circ \varphi(z)\|_X\, \omega(z) \\ &= \sup_{z\in\mathbb{D}} \|f(\varphi(z))\|_X v(\varphi(z)) \cdot \frac{\omega(z)}{v(\varphi(z))} \\ &\le \left(\sup_{z\in\mathbb{D}} \frac{\omega(z)}{v(\varphi(z))}\right) \|f\|_v. \end{aligned}$$

Thus, if (3.1) holds, then $\widetilde{C}_\varphi$ is bounded.

(b) The implication (i) ⇒ (ii) is immediate.

To show (ii) $\Rightarrow$ (iii): weak compactness of $\widetilde{C_\varphi}$ implies, by [19], that C_φ is weakly compact. Then, by [7, Corollary 8(b)], we have:

$$\lim_{n\to\infty} \frac{\|\varphi^n(z)\|_\omega}{\|\zeta^n\|_\upsilon} = 0,$$

which gives condition (iii). □

We now present an example to show that the three conditions in part (b) of Theorem 3.5 are not equivalent.

Example 3.6. *Consider the following cases:*

(1) Let $\omega(z) = \upsilon(z) = 1-|z|$, and $\varphi(z) = \frac{z}{2}$ for all $z \in \mathbb{D}$. These clearly satisfy the conditions of Theorem 3.5, and moreover:

$$\lim_{n\to\infty} \frac{\|e^{in\varphi(z)}\|_\upsilon}{\|e^{inz}\|_\upsilon} = 0.$$

However, the operator $\widetilde{C_\varphi} : H_\upsilon(\mathbb{D}, c_0) \to H_\omega(\mathbb{D}, c_0)$ is not weakly compact, since c_0 is not reflexive. This shows that condition (iii) does not imply (ii).

(2) Let $\upsilon(z) = 1-|z|^2$, $\omega(z) = (1-|z|^2)^2$, and let φ be the identity map on $\mathbb{D}$. Then:

$$\sup_{|\varphi(z)|>r} \frac{\omega(z)}{\upsilon(\varphi(z))} = \sup_{|z|>r} \frac{(1-|z|^2)^2}{1-|z|^2} = 1-r^2.$$

Thus,

$$\lim_{r\to 1^-} \sup_{|\varphi(z)|>r} \frac{\omega(z)}{\upsilon(\varphi(z))} = 0.$$

By [9, Theorem 3.3], the operator $C_\varphi : H_\upsilon(\mathbb{D}) \to H_\omega(\mathbb{D})$ is compact.

Now take $X = \ell^p$ for any $1 < p < \infty$. Since ℓ^p is reflexive, the operator

$$\widetilde{C_\varphi} : H_\upsilon(\mathbb{D}, \ell^p) \to H_\omega(\mathbb{D}, \ell^p)$$

is weakly compact, by [8, Theorem 17]. However, $\widetilde{C_\varphi}$ is not compact, since ℓ^p is infinite-dimensional. This shows that (ii) does not imply (i).

4. Vector-Valued Weighted Banach algebras

Let V be a countable family of weights. Bierstedt et al. in [4] defined and studied weighted spaces of holomorphic functions:

$$HV(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) : \|f\|_\upsilon = \sup_{z\in\mathbb{D}} |f(z)|\upsilon(z) < \infty, \quad \text{for all } \upsilon \in V \right\}.$$

This space is the direct limit of the family of Banach spaces $\{H_\upsilon(U)\}$, and thus forms a Fréchet space with seminorms $(\|\cdot\|_\upsilon)_{\upsilon\in V}$. For more details, see [16].

Let $U \subset \mathbb{C}$ be an open set. A subset $B \subset U$ is said to be *U-bounded* if it is bounded and the distance $d(B, \mathbb{C}\setminus U) > 0$. A family V of weights defined on U satisfies **Condition I** if for each U-bounded set B, there exists $\upsilon \in V$ such that $\inf_{z\in B} \upsilon(z) > 0$.

Carando and Sevilla-Peris in [13] provided a condition under which $HV(U)$ becomes an algebra, when V consists of radial, bounded weights.

Proposition 4.1 ([13], Proposition 1)**.** *Let U be an open and balanced subset of $\mathbb{C}$, and let V be a family of radial, bounded weights satisfying Condition I. Then $HV(U)$ is an algebra if for every $\upsilon \in V$, there exist $\omega \in V$ and $c > 0$ such that*

$$\upsilon(z) \le c\tilde{\omega}^2(z), \quad \text{for all } z \in U.$$

Corollary 4.2. *Assume that υ is a radial, bounded weight on $\mathbb{D}$. Then $H_\upsilon(\mathbb{D})$ is an algebra if and only if $H_\upsilon(\mathbb{D}) = H^\infty(\mathbb{D})$.*

Proof. Suppose $H_\upsilon(\mathbb{D})$ is an algebra. Then $H_{\tilde{\upsilon}}(\mathbb{D})$ is also an algebra. By the above proposition and the identity $\tilde{\tilde{\upsilon}} = \tilde{\upsilon}$, it follows that $\tilde{\upsilon}$ is bounded below. Hence,

$$H_{\tilde{\upsilon}}(\mathbb{D}) = H^\infty(\mathbb{D}).$$

The converse is straightforward to verify. □

Now, for a Banach algebra $\mathcal{A}$, we consider a weight υ (not necessarily bounded) and prove that $H_\upsilon(\mathbb{D}, \mathcal{A})$ forms a Banach algebra under appropriate assumptions.

Proposition 4.3. *Let $\mathcal{A}$ be a Banach algebra and let υ be a weight on $\mathbb{D}$ that is bounded below. Then $H_\upsilon(\mathbb{D}, \mathcal{A})$ is a Banach algebra.*

Proof. Since υ is bounded below, there exists $d > 0$ such that $\upsilon(z) \ge d$ for all $z \in \mathbb{D}$. Let $f, g \in H_\upsilon(\mathbb{D}, \mathcal{A})$. Then:

$$\begin{aligned} \|fg\|_\upsilon &= \sup_{z\in\mathbb{D}} \|f(z)g(z)\|_{\mathcal{A}}\upsilon(z) \\ &\le \sup_{z\in\mathbb{D}} \|f(z)\|_{\mathcal{A}}\|g(z)\|_{\mathcal{A}}\frac{\upsilon^2(z)}{d} \\ &\le \frac{1}{d}\|f\|_\upsilon\|g\|_\upsilon, \end{aligned} \tag{4.1}$$

so with the equivalent norm $\|f\| := \frac{1}{d}\|f\|_\upsilon$, the space $H_\upsilon(\mathbb{D}, \mathcal{A})$ is a normed algebra. Since it is also a Banach space, it follows that it is a Banach algebra. □

This behavior clearly occurs for certain weights, such as

$$\upsilon(z) = \frac{1}{(1-|z|)^p} \quad \text{for } p > 0, \quad \text{and} \quad \omega(z) = \frac{1}{1-e^{-|z|}} \quad \text{on } \mathbb{D}.$$

We can now obtain more detailed information about the Banach algebra $H_\upsilon(U, \mathcal{A})$. Before proceeding, observe the following lemma. The closed subspace $H^0_\upsilon(U) \subset H_\upsilon(U)$ is defined by

$$H^0_\upsilon(U) := \left\{ f \in H_\upsilon(U) : \lim_{|z|\to\partial U} |f(z)|\upsilon(z) = 0 \right\},$$

where $U \subset \mathbb{C}$ is open with boundary ∂U. The limit $|z| \to \partial U$ means that for each $\varepsilon > 0$, there exists a compact subset $K \subset U$ such that $|f(z)|\upsilon(z) < \varepsilon$ for all $z \in U \setminus K$.

Lemma 4.4. *Let $U \subset \mathbb{C}$ be an open connected set, and let υ be a strictly positive continuous weight on U. For any distinct points $z \ne w \in U$, either $\dim H^0_\upsilon(U) \le 1$, or there exists $f \in H^0_\upsilon(U)$ such that $f(z) \ne f(w)$. In particular, if $\dim H^0_\upsilon(U) \ge 2$, then $H^0_\upsilon(U)$ separates the points of U. [12, Lemma 4]*

Proposition 4.5. *Let υ be a strictly positive continuous weight on $\mathbb{D}$, and let $\mathcal{A}$ be a Banach algebra. If both $H_\upsilon(\mathbb{D})$ and $H_\upsilon(\mathbb{D}, \mathcal{A})$ are Banach algebras, then:*

(i) *$H_\upsilon(\mathbb{D}, \mathcal{A})$ is commutative if and only if $\mathcal{A}$ is commutative.*
(ii) *$H_\upsilon(\mathbb{D}, \mathcal{A})$ is unital if and only if $\mathcal{A}$ is unital and υ is bounded.*

(iii) *If $\mathcal{A}$ is a unital, commutative, and semisimple Banach algebra and υ is bounded, then $H_\upsilon(\mathbb{D},\mathcal{A})$ is also unital, commutative, and semisimple.*

Proof. (i) This is immediate from the inclusion $\mathcal{A}\otimes H_\upsilon(\mathbb{D})\subset H_\upsilon(\mathbb{D},\mathcal{A})$.

(ii) For the "only if" direction, suppose there exists a unit $\alpha\in H_\upsilon(\mathbb{D},\mathcal{A})$, i.e., $f\alpha=f$ for all $f\in H_\upsilon(\mathbb{D},\mathcal{A})$. In particular, for $f=g\otimes a$ with $g\in H_\upsilon(\mathbb{D})$ and $a\in\mathcal{A}$, we have

$$(g\otimes a)\alpha(z)=g(z)a\alpha(z)=g(z)a \quad \text{for all } z\in\mathbb{D}.$$

By Lemma 4.4, $H_\upsilon(\mathbb{D})$ separates points, so we can choose g such that $g(z)\neq 0$. It follows that $a\alpha(z)=a$, hence $\alpha(z)=e$, the unit in $\mathcal{A}$. Thus, α is the constant function e, and so

$$\|\alpha\|_\upsilon=\sup_{z\in\mathbb{D}}\|\alpha(z)\|\upsilon(z)=\|e\|\sup_{z\in\mathbb{D}}\upsilon(z)<\infty.$$

Hence, υ must be bounded. The converse is straightforward.

(iii) Suppose $f\in\operatorname{rad}(H_\upsilon(\mathbb{D},\mathcal{A}))$, the Jacobson radical. Then $1-fg$ is invertible for all $g\in H_\upsilon(\mathbb{D},\mathcal{A})$, i.e., there exists $h\in H_\upsilon(\mathbb{D},\mathcal{A})$ such that

$$(1-fg)h=1 \quad \text{in } H_\upsilon(\mathbb{D},\mathcal{A}),$$

which implies

$$(e-f(z)g(z))h(z)=e \quad \text{for all } z\in\mathbb{D}.$$

Fix $z_0\in\mathbb{D}$. By Lemma 4.4, there exists $k_0\in H_\upsilon(\mathbb{D})$ such that $k_0(z_0)=1$. Then, for any $a\in\mathcal{A}$, define $g=k_0\otimes a\in H_\upsilon(\mathbb{D},\mathcal{A})$, and at z_0 we get

$$(e-f(z_0)a)h(z_0)=e.$$

Thus $e-f(z_0)a$ is invertible in $\mathcal{A}$ for all $a\in\mathcal{A}$, implying $f(z_0)\in\operatorname{rad}(\mathcal{A})=\{0\}$, since $\mathcal{A}$ is semisimple. Hence, $f(z_0)=0$, and since z_0 was arbitrary, $f=0$. Therefore, $\operatorname{rad}(H_\upsilon(\mathbb{D},\mathcal{A}))=\{0\}$, showing semisimplicity. □

Conclusion

We established an isomorphism between the weighted Banach space $H_\upsilon^{2\pi}(\mathbb{G},X)$ of vector-valued holomorphic 2π-periodic functions on the upper half-plane and the weighted space $H_{\hat{\upsilon}}(\mathbb{D},X)$ on the unit disc. The construction is based on an explicit transformation of weights, associating a standard weight on $\mathbb{G}$ with a radial weight on $\mathbb{D}$ vanishing at the boundary. This identification allows the transfer of structural and operator-theoretic properties between the two settings.

References

[1] Abramovich, Y.A., & Aliprantis, C.D. (2002). An Invitation to Operator Theory. *Graduate Studies in Mathematics*, Vol. 50–51. American Mathematical Society.

[2] Ardalani, M.A., & Lusky, W. (2011). Weighted spaces of holomorphic 2π-periodic functions on the upper half-plane. *Functiones et Approximatio Commentarii Mathematici*, 44(2), 191–201.

[3] Arendt, W., Batty, C.J.K., Hieber, M., & Neubrander, F. (2001). Vector-Valued Laplace Transforms and Cauchy Problems. *Monographs in Mathematics*, Vol. 96. Birkhäuser.

[4] Bierstedt, K.D., Bonet, J., & Galbis, A. (1993). Weighted spaces of holomorphic functions on balanced domains. *Michigan Mathematical Journal*, 40, 271–297.

[5] Bierstedt, K.D., Bonet, J., & Taskinen, J. (1998). Associated weights and spaces of holomorphic functions. *Studia Mathematica*, 127, 137–168.

[6] Bierstedt, K.D., & Holtmanns, S. (1999). An operator representation for weighted spaces of vector-valued holomorphic functions. *Results in Mathematics*, 36, 9–20.

[7] Bonet, J., Domański, P., & Lindström, M. (1999). Essential norm and weak compactness of composition operators on weighted Banach spaces of analytic functions. *Canadian Mathematical Bulletin*, 42(2), 139–148.

[8] Bonet, J., Domański, P., & Lindström, M. (2001). Weakly compact composition operators on analytic vector-valued function spaces. *Annales Academiae Scientiarum Fennicae. Mathematica*, 26, 233–248.

[9] Bonet, J., Domański, P., Lindström, M., & Taskinen, J. (1998). Composition operators between weighted Banach spaces of analytic functions. *Journal of the Australian Mathematical Society*, 64(1), 101–118.

[10] Bonet, J., & Friz, M. (2002). Weakly compact composition operators on locally convex spaces. *Mathematische Nachrichten*, 245(1), 26–44.

[11] Bonet, J., Gómez-Collado, M.C., Jornet, D., & Wolf, E. (2012). Operator-weighted composition operators between weighted spaces of vector-valued analytic functions. *Annales Academiae Scientiarum Fennicae. Mathematica*, 37, 319–338.

[12] Bonet, J., & Vogt, D. (1997). Weighted spaces of holomorphic functions and sequence spaces. *Note di Matematica*, 17, 87–97.

[13] Carando, D., & Sevilla-Peris, P. (2009). Spectra of weighted algebras of holomorphic functions. *Mathematische Zeitschrift*, 263(4), 887–902.

[14] Conway, J.B. (1973). Functions of One Complex Variable. *Springer-Verlag.*

[15] Conway, J.B. (1990). A Course in Functional Analysis. *Springer-Verlag.*

[16] García, D., Maestre, M., & Rueda, P. (2000). Weighted spaces of holomorphic functions on Banach spaces. *Studia Mathematica*, 138, 1–24.

[17] García, D., Maestre, M., & Sevilla-Peris, P. (2006). Weakly compact composition operators between weighted spaces. *Note di Matematica*, 25(1), 205–220.

[18] Harutyunyan, A., & Lusky, W. (2008). On the boundedness of the differentiation operator between weighted spaces of holomorphic functions. *Studia Mathematica*, 184(3), 233–247.

[19] Laitila, J., & Tylli, H.O. (2015). Composition operators on vector-valued analytic function spaces: a survey. Preprint.

[20] Laursen, K.B., & Neumann, M.M. (2000). An Introduction to Local Spectral Theory. *Oxford University Press.*

[21] Nordgren, E.A. (1968). Composition operators. *Canadian Journal of Mathematics*, 20, 442–449.

[22] Ryff, J.V. (1966). Subordinate h^p functions. *Duke Mathematical Journal*, 33(2), 347–354.

[23] Vasilescu, F.-H. (1979). Calcul funcţional analitic multidimensional. *Editura Academiei R.S.R.*

[24] Wolf, E. (2011). Composition followed by differentiation between weighted Banach spaces of holomorphic functions. *RACSAM*, 105(2), 315–322.