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On Some Mappings on Random Banach $*-$Algebras with Derivations

Alireza Janfada[1] ✉, **Ahmad Khoshab**[1]

1. Department of Mathematics, University of Birjand, Birjand, Iran.

Email: ajanfada@birjand.ac.ir (Corresponding Author), ahmad_khoshab@yahoo.com

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ABSTRACT

This paper investigates the stability of certain linear mappings on random Banach $*-$algebras with derivations. We establish several theorems concerning additive mappings and derivations in these spaces, extending classical results to the probabilistic setting. Our main contributions include proving the existence and uniqueness of additive mappings satisfying specific functional inequalities and demonstrating conditions under which such mappings become derivations.

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1. Introduction

The investigation of derivations in random Banach algebras represents an emerging and significant area within functional analysis and operator algebras. This field primarily focuses on the stability of random functional equations under conditions of uncertainty. The foundational concept of stability for functional equations was first introduced by Ulam in 1940 [15]. Subsequently, Hyers provided a fundamental result by establishing the stability of the additive functional equation in Banach spaces [7]. This work was later generalized by Rassias, who introduced a more flexible framework for functional approximations under variable bounds [11].

During the early 21st century, stability theory was extended to the study of approximate derivations in Banach algebras. Researchers examined whether mappings approximately satisfying the derivation condition $D(xy) \approx D(x)y + xD(y)$ could be approximated by exact derivations [5, 6, 10, 13, 14]. Concurrently, random Banach algebras were developed as probabilistic extensions of classical Banach algebras, providing a robust framework for modeling stochastic systems and phenomena involving uncertainty. These structures have since played a crucial role in stochastic analysis and probability theory.

The intersection of these two research directions has led to growing interest in random derivations within random Banach algebras. Recent studies have explored whether Hyers-Ulam stability properties hold in these probabilistic settings and whether approximate random derivations can be closely approximated by exact ones [1, 2, 9]. Contemporary applications of these results span diverse areas including stochastic analysis, quantum mechanics, and uncertainty quantification.

In what follows, we adopt the standard terminology, notation, and conventions of random normed space theory as presented in [8, 12].

2. Preliminaries and Basic Definitions

This section introduces the fundamental concepts and definitions required for our subsequent analysis. We begin with the basic structures of random normed spaces and gradually build up to random Banach $*-$algebras.

Definition 2.1 (Distribution Functions)**.** *Let $\triangle^+$ denote the space of distribution functions, defined as:*

$$\triangle^+ := \big\{F : \mathbb{R} \cup \{-\infty, \infty\} \to [0, 1] : F \text{ is left-continuous and non-decreasing on } \mathbb{R}, \\ F(0) = 0, \text{ and } F(+\infty) = 1\big\}.$$

The subset $D^+ \subseteq \triangle^+$ is given by:

$$D^+ = \left\{F \in \triangle^+ : \ell^- F(+\infty) = 1\right\},$$

where $\ell^- f(x)$ denotes the left limit of function f at point x.

The space $\triangle^+$ is partially ordered by pointwise comparison: $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. The maximal element in this ordering is the distribution function:

$$\varepsilon_0(t) = \begin{cases} 0, & t \leq 0, \\ 1, & t > 0. \end{cases}$$

Definition 2.2 (Triangular Norms)**.** *A* continuous triangular norm *(briefly, a t-norm) is a function $T : [0, 1] \times [0, 1] \to [0, 1]$ satisfying:*

(1) Commutativity and associativity: $T(a, b) = T(b, a)$ and $T(a, T(b, c)) = T(T(a, b), c)$;

(2) Continuity: T is continuous as a two-variable function;
(3) Boundary condition: $T(a,1) = a$ for all $a \in [0,1]$;
(4) Monotonicity: $T(a,b) \leq T(c,d)$ whenever $a \leq c$ and $b \leq d$.

Common examples of continuous t-norms include:

- The product t-norm: $T_p(a,b) = ab$.
- The minimum t-norm: $T_M(a,b) = \min(a,b)$.
- The Łukasiewicz t-norm: $T_L(a,b) = \max(a+b-1, 0)$.

For a t-norm T and a sequence $\{x_n\}$ in $[0,1]$, we define recursively:

$$T_{i=1}^{n} x_i = \begin{cases} x_1, & n = 1, \\ T(T_{i=1}^{n-1} x_i, x_n), & n \geq 2, \end{cases}$$

and denote $T_{i=n}^{\infty} x_i = T_{i=1}^{\infty} x_{n+i}$.

Definition 2.3 (Random Normed Spaces)**.** *A* random normed space *(RN-space) is a triple (X, Λ, T), where:*

- *X is a vector space over $\mathbb{R}$ or $\mathbb{C}$;*
- *T is a continuous t-norm;*
- *$\Lambda : X \to D^+$ (with Λ_x denoting $\Lambda(x)$) satisfies:*

(RN1) $\Lambda_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;
(RN2) $\Lambda_{\alpha x}(t) = \Lambda_x\left(\frac{t}{|\alpha|}\right)$ for all $x \in X$, $\alpha \neq 0$;
(RN3) $\Lambda_{x+y}(t+s) \geq T(\Lambda_x(t), \Lambda_y(s))$ for all $x, y \in X$, $t, s \geq 0$.

Definition 2.4 (Convergence and Completeness in RN-spaces)**.** *Let (X, Λ, T) be an RN-space.*

(1) A sequence $\{x_n\}$ in X converges *to $x \in X$ if for every $\varepsilon > 0$ and $\lambda > 0$, there exists $N \in \mathbb{N}$ such that $\Lambda_{x_n - x}(\varepsilon) > 1 - \lambda$ for all $n \geq N$.*
(2) A sequence $\{x_n\}$ in X is Cauchy *if for every $\varepsilon > 0$ and $\lambda > 0$, there exists $N \in \mathbb{N}$ such that $\Lambda_{x_n - x_m}(\varepsilon) > 1 - \lambda$ for all $n, m \geq N$, $n \geq m$.*
(3) An RN-space is complete *if every Cauchy sequence converges. A complete RN-space is called a* random Banach space.

Theorem 2.5 ([12])**.** *If (X, Λ, T) is an RN-space and $\{x_n\}$ converges to x, then $\lim_{n\to\infty} \Lambda_{x_n}(t) = \Lambda_x(t)$ almost everywhere.*

Definition 2.6 (Random Normed Algebras)**.** *A* random normed algebra *is a random normed space (X, Λ, T) with an additional algebraic structure satisfying:*

$$\Lambda_{xy}(ts) = \Lambda_x(t)\Lambda_y(s) \quad \textit{for all } x, y \in X,\ t, s > 0.$$

A complete random normed algebra is called a random Banach algebra.

Definition 2.7 (Random Banach $*-$Algebras)**.** *A* random Banach $*-$algebra *is a random Banach algebra (X, Λ, T) equipped with an involution $* : X \to X$ satisfying for all $x, y \in X$ and scalars a, b:*

- *$x^{**} = x$;*
- *$(ax + by)^* = \bar{a}x^* + \bar{b}y^*$;*
- *$(xy)^* = y^* x^*$.*

Definition 2.8 (Derivations on Random Banach $*-$Algebras)**.** *Let X be a random Banach $*-$algebra. A* derivation *on X is a mapping $\mu : X \to X$ satisfying:*

$$\mu(xy) = x\mu(y) + \mu(x)y \quad \textit{for all } x, y \in X.$$

3. Main Results

In this section, we present our main results concerning additive mappings and derivations on random Banach $*-$algebras. Throughout, we denote by $T_\varepsilon := \{e^{i\theta} : 0 \le \theta \le \varepsilon\}$ and by e the unit element when it exists.

Theorem 3.1. *Let (X, Λ, T) be a complex random Banach $*-$algebra and let $\xi : X^2 \to D^+$ (with $\xi_{x,y}$ denoting $\xi(x, y)$) be a mapping satisfying:*

$$\lim_{n\to\infty} \xi_{2^n x, 2^n y}(2^n t) = 1, \tag{3.1}$$

$$\lim_{n\to\infty} T_{\ell=1}^{\infty}\left(\xi_{2^{n+\ell}x, 2^{n+\ell}y}\left(2^{n+\ell}t\right)\right) = 1 \tag{3.2}$$

for all $x, y \in X$ and $t > 0$. Suppose $f : X \to X$ is a function satisfying the following inequalities for all $x, y \in X$ and $t > 0$:

$$\Lambda_{f(x+y)-f(x)-f(y)}(t) \ge \xi_{x,y}(t), \tag{3.3}$$

$$\Lambda_{f(xy^*+yx^*)-f(x)y^*-xf(y^*)-f(y)x^*-yf(x^*)}(t) \ge \xi_{x,y}(t). \tag{3.4}$$

Then there exists a unique additive mapping $F : X \to X$ such that:

$$F(xy^* + yx^*) = F(x)y^* + xF(y^*) + F(y)x^* + yF(x^*), \tag{3.5}$$

$$\Lambda_{f(x)-F(x)}(t) \ge T_{\ell=1}^{\infty}\left(\xi_{2^\ell x, 2^\ell x}\left(2^\ell t\right)\right), \tag{3.6}$$

$$x[F(y) - f(y)] = 0 \quad \text{for all } x, y \in X. \tag{3.7}$$

Proof. We divide the proof into several logical steps.
Step 1: Construction of the additive mapping. Setting $y = x$ in (3.3) yields:

$$\Lambda_{f(2x)-2f(x)}(t) \ge \xi_{x,x}(t) \quad \text{for all } x \in X,\ t > 0,$$

which implies:

$$\Lambda_{\frac{f(2x)}{2}-f(x)}(t) \ge \xi_{x,x}(2t).$$

Iterating this process, we obtain for any positive integer n:

$$\Lambda_{\frac{f(2^{n+1}x)}{2^{n+1}}-\frac{f(2^n x)}{2^n}}(t) \ge \xi_{2^n x, 2^n x}(2^{n+1}t).$$

By induction, we derive the following key inequality:

$$\Lambda_{\frac{f(2^n x)}{2^n}-f(x)}(t) \ge T_{\ell=1}^{n}\left(\xi_{2^\ell x, 2^\ell x}\left(2^\ell t\right)\right) \quad \text{for all } x \in X,\ t > 0. \tag{3.8}$$

Replacing x by $2^m x$ in (3.8) gives:

$$\Lambda_{\frac{f(2^{n+m}x)}{2^{n+m}}-\frac{f(2^m x)}{2^m}}(t) \ge T_{\ell=1}^{n}\left(\xi_{2^{\ell+m}x, 2^{\ell+m}x}\left(2^{\ell+m}t\right)\right).$$

From condition (3.2), the right-hand side tends to 1 as $n, m \to \infty$. Therefore, $\left\{\frac{f(2^n x)}{2^n}\right\}_{n=1}^{\infty}$ is a Cauchy sequence in the complete space X, hence convergent. Define:

$$F(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \quad \text{for all } x \in X.$$

Step 2: Additivity of F. Replacing x, y by $2^n x, 2^n y$ in (3.3) and dividing by 2^n yields:

$$\Lambda_{\frac{f(2^n(x+y))}{2^n}-\frac{f(2^n x)}{2^n}-\frac{f(2^n y)}{2^n}}(t) \ge \xi_{2^n x, 2^n y}(2^n t).$$

Taking the limit as $n \to \infty$ and using (3.1) gives $F(x + y) = F(x) + F(y)$.
Step 3: Proof of approximation inequality (3.6). Taking the limit as $n \to \infty$ in (3.8) yields (3.6).

Step 4: Uniqueness of F. Suppose F' is another additive mapping satisfying (3.6). Then for any $x \in X$ and $t > 0$:

$$\begin{aligned}\Lambda_{F(x)-F'(x)}(t) &= \Lambda_{F(2^n x)-F'(2^n x)}(2^n t)\\ &\geq T\left(\Lambda_{F(2^n x)-f(2^n x)}(2^{n-1}t), \Lambda_{f(2^n x)-F'(2^n x)}(2^{n-1}t)\right)\\ &\geq T\left(T_{\ell=1}^{\infty}\left(\xi_{2^{\ell+n}x,2^{\ell+n}x}\left(2^{n+\ell-1}t\right)\right), T_{\ell=1}^{\infty}\left(\xi_{2^{\ell+n}x,2^{\ell+n}x}\left(2^{n+\ell-1}t\right)\right)\right).\end{aligned}$$

Letting $n \to \infty$ gives $\Lambda_{F(x)-F'(x)}(t) = 1$ for all $t > 0$, hence $F = F'$.
Step 5: Proof of property (3.5). Replacing x by $2^n x$ in (3.4) gives:

$$\Lambda_{f(2^n xy^*+2^n yx^*)-f(2^n x)y^*-2^n xf(y^*)-2^n f(y)x^*-yf(2^n x^*)}(t) \geq \xi_{2^n x,y}(t).$$

Dividing by 2^n and taking $n \to \infty$ using (3.1) and (3) yields:

(†) $$F(xy^* + yx^*) = F(x)y^* - xf(y^*) - f(y)x^* - yF(x^*).$$

Applying (†) to different arrangements gives:

$$\begin{aligned}2^n F(x)y^* + 2^n xf(y^*) + 2^n f(y)x^* + 2^n yF(x^*) &= F(2^n x \cdot y^* + y \cdot 2^n x^*)\\ &= F(x \cdot 2^n y^* + 2^n y \cdot x^*)\\ &= 2^n F(x)y^* + xf(2^n y^*) + f(2^n y)x^* + 2^n yF(x^*).\end{aligned}$$

Thus we obtain:

(‡) $$xF(y^*) + F(y)x^* = \lim_{n\to\infty}\left[x\frac{f(2^n y^*)}{2^n} + \frac{f(2^n y)}{2^n}x^*\right] = xf(y^*) + f(y)x^*.$$

Combining (†) and (‡) yields (3.5).
Step 6: Proof of annihilation property (3.7). Multiplying (‡) by i gives:

$$ixF(y^*) + iF(y)x^* = ixf(y^*) + if(y)x^*.$$

Replacing x by ix in (‡) yields:

$$ixF(y^*) - iF(y)x^* = ixf(y^*) - if(y)x^*.$$

Subtracting these two equations gives $2iF(y)x^* = 2if(y)x^*$, which simplifies to $x^*[F(y) - f(y)] = 0$ for all $x, y \in X$. Since the involution is bijective, this is equivalent to (3.7). □

Theorem 3.2. *Let (X, Λ, T) be a unital complex random Banach $*-$algebra and let $\xi : X^2 \to D^+$ satisfy the conditions of Theorem 3.1. Suppose $f : X \to X$ is a mapping such that for all $x, y \in X$, $\lambda \in T_\varepsilon$, and $t > 0$:*

$$\Lambda_{f(\lambda x+\lambda y)-\lambda f(x)-\lambda f(y)}(t) \geq \xi_{x,y}(t),$$

and inequality (3.4) holds. Then f is a derivation on X.

Proof. Taking $\lambda = 1$ in (3.2) shows that f satisfies the conditions of Theorem 3.1. Hence, there exists a unique additive mapping $F : X \to X$ satisfying (3.5), (3.6), and (3.7).
Linearity of F. Setting $y = x$ in (3.2) gives:

$$\Lambda_{f(2\lambda x)-2\lambda f(x)}(t) \geq \xi_{x,x}(t) \quad \text{for all } \lambda \in T_\varepsilon.$$

Replacing x by $2^n x$ and dividing by 2^{n+1} yields:

$$\Lambda_{\frac{f(2^{n+1}\lambda x)}{2^{n+1}}-\frac{\lambda f(2^n x)}{2^n}}(t) \geq \xi_{2^n x,2^n x}(2^{n+1}t).$$

Taking $n \to \infty$ and using (3) gives $F(\lambda x) = \lambda F(x)$ for all $\lambda \in T_\varepsilon$.

Identification of f and F. Since X is unital, setting $x = e$ in (3.7) gives $F(y) = f(y)$ for all $y \in X$. Thus $f = F$, and in particular, f satisfies (3.5):

$$(\star) \quad f(xy^* + yx^*) = f(x)y^* + xf(y^*) + f(y)x^* + yf(x^*).$$

Derivation property. Replacing y by iy in $(\star)$ yields:

$$-if(xy^*) + if(yx^*) = -if(x)y^* - ixf(y^*) + if(y)x^* + iyf(x^*).$$

Multiplying by i gives:

$$(\star\star) \quad f(xy^*) - f(yx^*) = f(x)y^* + xf(y^*) - f(y)x^* - yf(x^*).$$

Adding $(\star)$ and $(\star\star)$ and dividing by 2 yields:

$$f(xy^*) = f(x)y^* + xf(y^*).$$

Finally, setting $y = y^*$ gives the derivation property:

$$f(xy) = f(x)y + xf(y) \quad \text{for all } x, y \in X.$$

□

Corollary 3.3. *Let (X, Λ, T) be a semiprime unital complex random Banach $*-$algebra and let $\xi : X^2 \to D^+$ satisfy the conditions of Theorem 3.1. If $f : X \to X$ satisfies inequalities (3.3) and (3.4), then f is a derivation on X.*

Proof. Since X is unital, Theorem 3.1 applies, and setting $x = e$ in (3.7) gives $f = F$. In particular, f satisfies $(\star)$ from the previous proof.

Define for each $x \in X$:

$$D(x) := f(x^2) - f(x)x - xf(x).$$

Setting $y = x^*$ in $(\star)$ yields:

$$(1) \qquad D(x) + D(x^*) = 0 \quad \text{for all } x \in X.$$

Setting $y = xy^* + yx^*$ in $(\star)$ and simplifying gives:

$$f(x(y + y^*)x^*) = -D(x)y^* - yD(x^*) + f(x)(y + y^*)x^* + x(y + y^*)f(x^*) + xf(y + y^*)x^*.$$

Taking $y = y - y^*$ in this equation yields:

$$(2) \qquad D(x)(y - y^*) - (y - y^*)D(x^*) = 0 \quad \text{for all } x, y \in X.$$

Multiplying (2) by i gives one relation, while replacing y by iy in (2) gives another. Combining these yields:

$$(3) \qquad D(x)y = yD(x^*) \quad \text{for all } x, y \in X.$$

Setting $y = e$ in (3) gives $D(x) = D(x^*)$. Combining with (1) gives $D(x) = 0$ for all $x \in X$. Therefore:

$$f(x^2) = f(x)x + xf(x) \quad \text{for all } x \in X,$$

so f is a Jordan derivation. Since X is semiprime, every Jordan derivation is a derivation, completing the proof. □

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