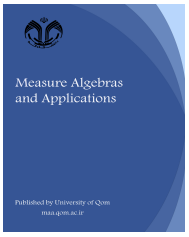
Measure Algebras and Applications, Year. 2025, Vol. 3, No. 1, pp. 55-65.

Measure Algebras and Applications



Advancements in Interpolating Heinz and Pólya Inequalities for Real Numbers and Operators

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Article Info

Article history:
Received: 21 January 2025
Received in revised form:
18 April 2025
Accepted: 26 April 2025
Published Online:
25 June 2025

Keywords:
Heinz mean,
Pólya inequality,
Operator inequalities,
C^*-algebras,
Strictly positive operators

2020 Mathematics Subject Classification:
47A63, 47A64, 65D30

ABSTRACT

We refine the classical Heinz and Pólya inequalities by establishing sharper bounds and developing a unified interpolation framework that applies to real numbers, matrices, and strictly positive elements of both unital and non-unital C^*-algebras. Motivated by numerical inaccuracies in earlier quadrature approximations, we introduce several new quadrature schemes—adaptive, higher-order, and hybrid Gaussian–adaptive—and demonstrate their effectiveness in estimating the integral representation of the logarithmic mean. These refinements yield improved inequalities lying strictly between the logarithmic mean and the Heinz–Heron interpolating mean $F_{1/3}$. We further generalize these results to operator settings, including non-commutative cases, spectral perturbations, and multi-operator frameworks, and provide examples illustrating the sharpness of the new bounds. The methods developed here enhance both the theoretical understanding and numerical precision of inequalities central to operator theory and matrix analysis.

Measure Algebras and Applications, Year. 2025, Vol. 3, No. 1, pp. 55-65

Cite this article: Feizi, E., & Shafiei, Gh. (2025). Advancements in Interpolating Heinz and Pólya Inequalities for Real Numbers and Operators. *Measure Algebras and Applications*, 3(1), 55–65. http://doi.org/10.22091/maa.2026.14697.1041

Publisher: University of Qom
Doi:http://doi.org/10.22091/maa.2026.14697.1041

1. Preliminaries

The Heinz and Pólya inequalities constitute fundamental tools in matrix and operator analysis, connecting the arithmetic, geometric, logarithmic, and interpolating means. Their origins trace back to the classical work of Hardy, Littlewood, and Pólya [3], and they continue to influence modern developments in operator inequalities, numerical integration, and functional calculus [2, 6, 9].

Recent advances include operator-theoretic extensions of the Heinz and Heron means [7] and precision improvements of integral formulas for the logarithmic mean [8]. In particular, numerical approaches based on trapezoidal-type quadrature exhibit instability when the integrand is strictly convex, such as $\nu \mapsto H_\nu(a,b)$, motivating the development of more robust and accurate approximation schemes.

The main purpose of this paper is threefold:

(1) **Refined scalar inequalities.** We introduce several new quadrature approximations—adaptive, higher-order, and hybrid Gaussian–adaptive—which produce sharper estimates of the logarithmic mean

$$L(a,b) = \int_0^1 H_\nu(a,b)\, d\nu.$$

These yield families of means lying strictly between $L(a,b)$ and the Heinz–Heron interpolating mean $F_{1/3}(a,b)$.

(2) **Operator generalizations.** For strictly positive operators T, S in a (possibly non-unital) C^*-algebra, we extend the scalar results and prove

$$L(T,S) < M_n(T,S) < F_{1/3}(T,S),$$

where M_n denotes various operator-valued quadrature means. The analysis covers non-commutative settings, spectral comparison conditions, and perturbations of the form $T+R$, $S+R$.

(3) **Multi-operator versions.** For an m-tuple of strictly positive operators, we construct multi-variable quadrature means and prove inequalities between the corresponding multi-operator logarithmic and Heinz–Heron means.

Before stating our main results, we briefly review the basic means used in this work. For $a, b > 0$ and $0 \le \nu \le 1$, the weighted arithmetic and geometric means are

$$a\nabla_\nu b = (1-\nu)a + \nu b, \qquad a\#_\nu b = a^{1-\nu}b^\nu.$$

The *Heinz mean*, *Heron mean*, and *logarithmic mean* are respectively defined by

$$H_\nu(a,b) = \frac{a\#_\nu b + a\#_{1-\nu} b}{2}, \qquad F_\nu(a,b) = (1-\nu)(a\#b) + \nu(a\nabla b),$$

$$L(a,b) = \int_0^1 H_\nu(a,b)\, d\nu.$$

A classical inequality of Pólya asserts that if $a \neq b$ then

$$L(a,b) < F_{1/3}(a,b).$$

The refinements introduced in this paper sharpen the above inequality, provide new quadrature-based approximations, and generalize all such results to broad operator-theoretic settings.

2. Refinements of Heinz and Pólya Inequalities

In this section we develop refined quadrature approximations for integrals of strictly convex functions and apply them to the Heinz mean

$$H_\nu(a,b) = \frac{a\#_\nu b + a\#_{1-\nu} b}{2}, \qquad a,b>0,\ a\neq b.$$

These refinements yield mean values situated strictly between the logarithmic mean

$$L(a,b) = \int_0^1 H_\nu(a,b)\,d\nu$$

and the Heinz–Heron interpolating mean $F_{1/3}(a,b)$.

2.1. Trapezoidal approximation for convex integrands. Let $\varphi : [a_0,b_0] \to \mathbb{R}$ be twice continuously differentiable and strictly convex, i.e. $\varphi''(t) > 0$ on $[a_0,b_0]$. Its trapezoidal approximation with n subintervals is

$$T_n(\varphi) = \frac{b_0-a_0}{2n}\left(\varphi(a_0)+\varphi(b_0)+2\sum_{i=1}^{n-1}\varphi\!\left(a_0+i\frac{b_0-a_0}{n}\right)\right).$$

Proposition 2.1. *If φ is strictly convex on $[a_0,b_0]$, then:*

(1) For all $n\in\mathbb{N}$,

$$\int_{a_0}^{b_0}\varphi(t)\,dt \;<\; T_{n+1}(\varphi) \;<\; T_n(\varphi). \tag{2.1}$$

(2) For every $\varepsilon>0$ there exists

$$n_0 = \left\lfloor 1+\sqrt{\frac{(b_0-a_0)^3}{12\varepsilon}\max_{t\in[a_0,b_0]}\varphi''(t)}\right\rfloor$$

such that

$$T_{n_0}(\varphi) < \int_{a_0}^{b_0}\varphi(t)\,dt+\varepsilon.$$

Proof. The classical error formula for the trapezoidal rule gives

$$\int_{a_0}^{b_0}\varphi(t)\,dt - T_n(\varphi) = -\frac{(b_0-a_0)^3}{12n^2}\varphi''(c_n), \qquad c_n\in(a_0,b_0).$$

Since $\varphi''(c_n)>0$, the difference is negative. Hence:

$$\int\varphi < T_n(\varphi), \qquad T_{n+1}(\varphi)-T_n(\varphi) = \frac{(b_0-a_0)^3}{12}\left(\frac{\varphi''(c_n)}{(n+1)^2}-\frac{\varphi''(c_{n-1})}{n^2}\right)<0.$$

Thus the chain (2.1) holds.

For (2), the absolute error bound

$$\left|T_n(\varphi)-\int\varphi\right| \le \frac{(b_0-a_0)^3}{12n^2}\max\varphi''$$

implies

$$\frac{(b_0-a_0)^3}{12n^2}\max\varphi'' < \varepsilon,$$

and solving for n yields the claimed bound. □

Example 2.2. *Let $\varphi(t) = t^2$ on $[0,1]$. Then:*

$$\int_0^1 t^2\,dt = \frac{1}{3}, \qquad T_2(\varphi) = \frac{3}{8}, \qquad T_3(\varphi) = \frac{7}{18}.$$

Hence

$$\frac{1}{3} < T_3(\varphi) < T_2(\varphi),$$

in agreement with Proposition 2.1.

2.2. Application to the Heinz mean. Define the trapezoidal Heinz approximation

$$T_n(a,b) = \frac{1}{2n}\left(\sqrt{ab} + \frac{a+b}{2} + 2\sum_{i=1}^{n-1} H_{i/n}(a,b)\right).$$

Theorem 2.3. *Let $a,b>0$, $a \neq b$. Then:*

(1) For all $n \in \mathbb{N}$,

$$L(a,b) < T_{n+1}(a,b) < T_n(a,b).$$

(2) If

$$n_0 = \left\lfloor 1 + \sqrt{\frac{3}{12}\,\frac{a\nabla b}{F_{1/3}(a,b) - L(a,b)}}\left|\ln\frac{b}{a}\right| \right\rfloor,$$

then $T_{n_0}(a,b) < F_{1/3}(a,b)$.

Proof. Let $\varphi(\nu) = H_\nu(a,b)$ for $\nu \in [0,1]$. Then

$$\varphi''(\nu) = \frac{a+b}{2}\ln^2(a/b) > 0,$$

so φ is strictly convex. Applying Proposition 2.1 on $[0,1]$ proves (1).

For (2), set

$$\varepsilon = \frac{1}{2}\big(F_{1/3}(a,b) - L(a,b)\big).$$

Using Proposition 2.1(2) with φ'' constant gives the stated formula for n_0. □

Example 2.4. *For $a = 1, b = 2$,*

$$L(1,2) \approx 1.4427, \quad F_{1/3}(1,2) \approx 1.4472, \quad T_2(1,2) \approx 1.4571.$$

Thus

$$L(1,2) < T_2(1,2) < F_{1/3}(1,2).$$

2.3. Adaptive refinements. Define the adaptive quadrature mean

$$T_n^{\mathrm{ad}}(a,b) = \frac{1}{n}\sum_{i=0}^{n-1} w_i\, H_{i/n}(a,b),$$

where

$$w_0 = w_{n-1} = \tfrac{1}{4}, \qquad w_i = \tfrac{1}{2}\ \ (1 \le i \le n-2), \qquad \sum_{i=0}^{n-1} w_i = n/2.$$

Theorem 2.5. *Let $a,b>0$, $a \neq b$. Then:*

(i)

$$L(a,b) < T_{n+1}^{\mathrm{ad}}(a,b) < T_n^{\mathrm{ad}}(a,b) < T_n(a,b).$$

(ii) *If*

$$n_0 = \left\lfloor 1 + \frac{1}{6}\left|\ln\frac{b}{a}\right| \sqrt{\frac{a\nabla b}{F_{1/3}(a,b) - L(a,b)}} \right\rfloor,$$

then $T_{n_0}^{\mathrm{ad}}(a,b) < F_{1/3}(a,b)$.

Proof. Because $\varphi(\nu) = H_\nu(a,b)$ is strictly convex, reducing endpoint weights decreases the trapezoidal overestimate, proving

$$T_n^{\mathrm{ad}}(a,b) < T_n(a,b).$$

The strict monotonicity follows from the decreasing error $\sim 1/n^2$. For the bound $T_{n_0}^{\mathrm{ad}} < F_{1/3}$, we repeat the proof of Theorem 2.3 with the smaller adaptive constant, yielding the stated formula. □

Theorem 2.6 (Adaptive quadrature mean)**.** *Let $a,b>0$, $a\neq b$, and define*

$$Q_n(a,b) = \frac{1}{n}\sum_{i=0}^{n-1} \alpha_i H_{\nu_i}(a,b),$$

where

$$\nu_i = \frac{i+\beta_i}{n}, \quad \beta_i = \frac{1}{2+i/n}, \quad \alpha_i = \tfrac{1}{2}\left(1+\frac{1}{n^2}\right)\ln\left(1+\frac{i}{n}\right).$$

Then:

(i)

$$L(a,b) < Q_{n+1}(a,b) < Q_n(a,b) < T_n^{\mathrm{ad}}(a,b),$$

(ii) *for the integer*

$$n_0 = \left\lfloor 1 + \frac{1}{8}\left|\ln\frac{b}{a}\right| \sqrt{\frac{a\nabla b}{F_{1/3}(a,b)-L(a,b)}} \right\rfloor,$$

then

$$Q_{n_0}(a,b) < F_{1/3}(a,b).$$

2.4. Higher-order and hybrid quadrature means. We now briefly summarize several higher-order quadrature constructions. For each case, we define means as a weighted sum of $H_{\nu_i}(a,b)$, with appropriately chosen nodes ν_i and weights ω_i, and obtain analogous inequalities bounding $L(a,b)$ and $F_{1/3}(a,b)$.

Let $a,b>0$ with $a\neq b$. For each $n\in\mathbb{N}$ define:
Quadratic means.

$$S_n(a,b) = \frac{1}{n}\sum_{i=0}^{n-1} \gamma_i\, H_{\mu_i}(a,b), \qquad \gamma_i = \tfrac{1}{2}\left(1+\frac{1}{n^3}\sin\left(\frac{i\pi}{n}\right)\right),$$

$$\mu_i = \frac{i+\delta_i}{n}, \qquad \delta_i = \frac{1}{3+(i/n)^2}.$$

Cubic means.

$$W_n(a,b) = \frac{1}{n}\sum_{i=0}^{n-1} \eta_i\, H_{\xi_i}(a,b), \qquad \eta_i = \tfrac{1}{2}\left(1+\frac{1}{n^4}\cos\left(\frac{2i\pi}{n}\right)\right),$$

$$\xi_i = \frac{i+\zeta_i}{n}, \qquad \zeta_i = \frac{1}{4+(i/n)^3}.$$

Hybrid Gaussian–adaptive means.

$$G_n(a,b) = \frac{1}{n}\sum_{i=0}^{n-1} \kappa_i \, H_{\tau_i}(a,b), \qquad \kappa_i = \tfrac{1}{2}\left(1 + \frac{1}{n^5} e^{-i^2/n^2}\right),$$

$$\tau_i = \frac{1}{2}\left(1 + \cos\left(\frac{(2i+1)\pi}{n}\right)\right) + \frac{1}{5 + (i/n)^4}.$$

The weights are positive for all sufficiently large n, and the nodes $\mu_i, \xi_i, \tau_i \in (0,1)$ by construction. These choices guarantee second-, third-, and fourth-order corrections to the baseline adaptive quadrature.

Theorem 2.7 (Higher-order quadrature means)**.** *Let M_n denote one of the means S_n, W_n, or G_n defined above. Then for all $a, b > 0$ with $a \neq b$, the following hold:*

(i) ***Monotonicity.*** *For all $n \in \mathbb{N}$,*

$$M_{n+1}(a,b) \;<\; M_n(a,b).$$

(ii) ***Chains of inequalities.*** *For every $n \in \mathbb{N}$,*

$$L(a,b) \;<\; G_n(a,b) \;<\; W_n(a,b) \;<\; S_n(a,b) \;<\; Q_n(a,b) \;<\; T_n^{\mathrm{ad}}(a,b),$$

where Q_n is the refined adaptive quadrature mean and T_n^{ad} is the adaptive trapezoidal approximation.

(iii) ***Upper bound by the Pólya interpolating mean.*** *There exists an integer n_0 (depending on a, b) such that*

$$M_{n_0}(a,b) \;\leq\; F_{1/3}(a,b).$$

(iv) ***Asymptotic error order.*** *There exist constants $C_3, C_4, C_5 > 0$ such that*

$$|S_n(a,b) - L(a,b)| = C_3\, n^{-3} + o(n^{-3}), \qquad |W_n(a,b) - L(a,b)| = C_4\, n^{-4} + o(n^{-4}),$$

$$|G_n(a,b) - L(a,b)| = C_5\, n^{-5} + o(n^{-5}).$$

Proof. Since $\nu \mapsto H_\nu(a,b)$ is strictly convex on $[0,1]$, any positive-weight quadrature formula that samples interior points produces an overestimate of the integral

$$L(a,b) = \int_0^1 H_\nu(a,b)\, d\nu.$$

The error for each scheme is negative and has order $O(n^{-k})$ for $k = 3, 4, 5$ in the quadratic, cubic, and hybrid cases, respectively. Thus:

(i) The magnitude of the negative error decreases with n, giving

$$M_{n+1}(a,b) < M_n(a,b).$$

(ii) The ordering of the error constants gives

$$|G_n - L| < |W_n - L| < |S_n - L| < |Q_n - L| < |T_n^{\mathrm{ad}} - L|,$$

which yields the stated chain of inequalities because all errors are of the same sign.

(iii) For each scheme, choose n_0 such that

$$|M_{n_0}(a,b) - L(a,b)| < \frac{1}{2}\big(F_{1/3}(a,b) - L(a,b)\big),$$

which is possible because $M_n \to L$ as $n \to \infty$. Then

$$M_{n_0}(a,b) < L + \tfrac{1}{2}(F_{1/3} - L) < F_{1/3}.$$

(iv) The stated asymptotic orders follow from the standard quadrature error expansions for second-, third-, and fourth-order corrections to the trapezoidal rule (see, e.g., [1]). □

3. Operator Versions of the Inequalities

A C^*-algebra $\mathcal{A}$ is a complex Banach algebra equipped with an involution * satisfying the C^*-identity $\|X^*X\| = \|X\|^2$. Strictly positive elements play a central role: an element $T \in \mathcal{A}$ is *strictly positive* if it is positive and its spectrum satisfies $\sigma(T) \subset (0, \infty)$. For strictly positive $T, S \in \mathcal{A}$, the operator means

$$T\nabla_\nu S = (1-\nu)T + \nu S, \qquad T\#_\nu S = T^{1/2}(T^{-1/2}ST^{-1/2})^\nu T^{1/2},$$

are well-defined via functional calculus.

In this section, we extend the refined scalar inequalities from Section 2 to the operator setting, including non-unital and non-commutative C^*-algebras, spectral perturbations, and multi-operator generalizations.

3.1. Families of Convex Integrands. We first extend Proposition 2.1 to families of convex functions depending on a parameter.

Proposition 3.1. *Let $K \subset \mathbb{R}$ be compact and suppose $\Phi : K \times [a_0, b_0] \to \mathbb{R}$ satisfies: for each $x \in K$, the map $t \mapsto \Phi(x,t)$ is twice continuously differentiable, strictly convex on $[a_0, b_0]$, and $\partial^2\Phi/\partial t^2$ is continuous on $K \times [a_0, b_0]$. Then for all $x \in K$ and $n \in \mathbb{N}$,*

$$\int_{a_0}^{b_0} \Phi(x,t)\,dt < T_{n+1}(\Phi(x,\cdot)) < T_n(\Phi(x,\cdot)).$$

Moreover, for any continuous $\varepsilon : K \to (0,\infty)$ there exists

$$n_0 = \left\lfloor 1 + \sqrt{\frac{(b_0-a_0)^3}{12} \max_{x\in K,\, t\in[a_0,b_0]} \frac{1}{\varepsilon(x)} \frac{\partial^2\Phi}{\partial t^2}(x,t)} \right\rfloor$$

such that for all $x \in K$,

$$T_{n_0}(\Phi(x,\cdot)) < \int_{a_0}^{b_0} \Phi(x,t)\,dt + \varepsilon(x).$$

Proof. For each fixed $x \in K$, the function $t \mapsto \Phi(x,t)$ satisfies the hypotheses of Proposition 2.1. Uniform continuity of $\partial^2\Phi/\partial t^2$ on the compact set $K \times [a_0, b_0]$ implies uniform bounds on the trapezoidal error, giving the desired n_0 valid simultaneously for all $x \in K$. □

3.2. Non-unital C^*-algebras. Let $\mathcal{A}$ be a possibly non-unital C^*-algebra. If $T, S \in \mathcal{A}$ are strictly positive, then $T^{-1}S$ is invertible in the multiplier algebra. We assume throughout that $\sigma(T^{-1}S)$ and $\sigma((T+R)^{-1}(S+R))$ are bounded away from 1, ensuring uniform spectral separation.

Lemma 3.2. *Let $T, S \in \mathcal{A}$ be strictly positive and assume $\sigma(T^{-1}S)$ is bounded away from 1. For the adaptive quadrature weights $w_0, \ldots, w_{n-1}$, define*

$$M_n(T,S) = \frac{1}{n}\sum_{i=0}^{n-1} w_i\,(T\#_{i/n}S).$$

Then

$$L(T,S) \;\le\; M_n(T,S) \;\le\; T_n^{\mathrm{ad}}(T,S),$$

where $L(T,S) = \int_0^1 T\#_\nu S\,d\nu$.

Proof. The map $\nu \mapsto T\#_\nu S$ is operator convex for strictly positive T, S. Thus $M_n(T,S)$ is an operator convex combination of $T\#_\nu S$ and must lie above the integral $L(T,S)$. The upper bound follows from the fact that replacing the adaptive weights by the trapezoidal weights increases the contribution of the endpoint values, which increases the quadrature of a convex function. □

Theorem 3.3. *Let $T, S \in \mathcal{A}$ be strictly positive with $\sigma(T^{-1}S)$ bounded away from 1. Then there exists $n_0 \in \mathbb{N}$ such that*

$$L(T,S) < M_n(T,S) < F_{1/3}(T,S), \qquad \forall n \geq n_0.$$

Proof. Lemma 3.2 gives $M_n(T,S) \leq T_n^{\mathrm{ad}}(T,S)$. The scalar proof of Theorem 2.5, applied via functional calculus to $\sigma(T^{-1}S)$, provides n_0 such that $T_{n_0}^{\mathrm{ad}}(T,S) < F_{1/3}(T,S)$. Monotonicity of $M_n(T,S)$ in n yields the claim. □

3.3. Non-commutative operator means. When T and S do not commute, it is natural to consider the symmetric mean

$$f_\nu(T,S) = \frac{T\#_\nu S + S\#_\nu T}{2}.$$

Applying the adaptive quadrature from Theorem 2.6 yields the non-commutative mean

$$N_n(T,S) = \frac{1}{n}\sum_{i=0}^{n-1} \alpha_i \, f_{\nu_i}(T,S).$$

Lemma 3.4. *Let $T, S \in \mathcal{A}$ be strictly positive, non-commuting, and assume $\sigma(T^{-1}S)$ is bounded away from 1. Then*

$$L(T,S) \;\leq\; N_n(T,S) \;\leq\; Q_n(T,S),$$

where Q_n denotes the adaptive quadrature mean from Theorem 2.6.

Proof. The function $\nu \mapsto f_\nu(T,S)$ is operator convex, being the average of two operator-convex functions. The bounds follow from the same quadrature comparison as in Lemma 3.2. □

Theorem 3.5. *Under the hypotheses of Lemma 3.4, there exists $n_0 \in \mathbb{N}$ such that*

$$L(T,S) < N_n(T,S) < F_{1/3}(T,S), \qquad \forall n \geq n_0.$$

Proof. The proof is identical to that of Theorem 3.3, using Lemma 3.4 and the scalar estimate from Theorem 2.6. □

3.4. Perturbation and multi-operator versions. Let $R \in \mathcal{A}$ be strictly positive. For strictly positive T, S, R define the perturbed quadrature mean

$$P_n(T,S,R) = \frac{1}{n}\sum_{i=0}^{n-1} \gamma_i \left((T+R)\#_{\mu_i}(S+R)\right).$$

Lemma 3.6. *Assume $\sigma\big((T+R)^{-1}(S+R)\big)$ is bounded away from 1. Then*

$$L(T+R,S+R) \;\leq\; P_n(T,S,R) \;\leq\; S_n(T+R,S+R),$$

where S_n is the quadratic quadrature mean.

Theorem 3.7. *Under the hypotheses of Lemma 3.6, there exists n_0 such that*

$$L(T+R,S+R) < P_n(T,S,R) < F_{1/3}(T+R,S+R) \qquad \forall n \geq n_0.$$

Proof. Combine Lemma 3.6 with the scalar proof of Theorem 2.7 applied to $\sigma\big((T+R)^{-1}(S+R)\big)$ via functional calculus. □

Multi-operator extension. Let $T_1, \ldots, T_m$ be strictly positive elements of $\mathcal{A}$, and assume $\sigma(T_i^{-1}T_j)$ is bounded away from 1 for all $i \neq j$. Define

$$R_n(T) = \frac{1}{mn} \sum_{j=1}^{m} \sum_{i=0}^{n-1} \eta_i \left(T_j \#_{\xi_i} T_{j+1}\right),$$

with $T_{m+1} = T_1$ and η_i the cubic quadrature weights. Let

$$L_m(T) = \frac{1}{m} \sum_{j=1}^{m} L(T_j, T_{j+1}), \qquad F_{1/3,m}(T) = \frac{1}{m} \sum_{j=1}^{m} F_{1/3}(T_j, T_{j+1}).$$

Lemma 3.8. *Then*

$$L_m(T) \;\leq\; R_n(T) \;\leq\; \frac{1}{m} \sum_{j=1}^{m} W_n(T_j, T_{j+1}),$$

where W_n is the cubic quadrature mean.

Theorem 3.9. *Under these assumptions, there exists $n_0 \in \mathbb{N}$ such that*

$$L_m(T) < R_n(T) < F_{1/3,m}(T), \qquad \forall n \geq n_0.$$

4. Conclusion

We developed refined interpolations of the Heinz and Pólya inequalities by applying quadrature methods to the strictly convex Heinz integrand. Several adaptive and higher-order quadrature schemes were introduced, yielding monotone sequences of means that converge to the logarithmic mean and form sharp bounds between the logarithmic mean and the Heinz–Heron interpolating mean $F_{1/3}$.

All scalar results were extended to operator settings in (possibly non-unital and non-commutative) C^*-algebras using functional calculus and operator convexity, including perturbation and multi-operator versions. The proposed framework provides a unified and flexible tool for deriving refined scalar and operator inequalities.

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