Measure Algebras and Applications, Year. 2025, Vol. 3, No. 1, pp. 43-54.

Measure Algebras and Applications

Quantum Topological Dynamical Systems: Foundations

Mehdi Akhavani[1], **Seyed Ebrahim Akrami**[2]✉

1. Department of Mathematics, Semnan University, Semnan, Iran.
   Email: m.akhavani@semnan.ac.ir
2. Corresponding Author, Department of Mathematics, Semnan University, Semnan, Iran, School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box: 19395-5746, Tehran, Iran and School of Physics, Institute for Research in Fundamental Sciences (IPM), P.O. Box: 19395-5531, Tehran, Iran. Email: akramisa@semnan.ac.ir

Article Info

Article history:
Received: 17 January 2025
Received in revised form: 14 April 2025
Accepted: 23 April 2025
Published Online: 25 June 2025

Keywords:
Quantum dynamics,
Topological dynamics,
Probability measure densities,
Continuity equation,
Cocycle

2020 Mathematics Subject Classification:
37A50, 37C25, 81P05

ABSTRACT

This paper introduces a framework for Quantum Topological Dynamical Systems (QTDS), reformulating classical topological dynamics in a quantum-inspired probabilistic context. The fundamental objects are maps or flows on spaces of probability densities over a manifold, rather than on the manifold itself. We provide a general construction for such dynamics from classical systems and analyze their properties. The central contribution is the introduction and characterization of microscopic fixed and periodic points and invariant sets, with necessary and sufficient conditions linking these probabilistic concepts to classical deterministic counterparts. This work establishes a dictionary for translating classical topological dynamics into the language of evolving probability densities.

Cite this article: Akhavani, M., & Akrami, S.E. (2025). Quantum Topological Dynamical Systems: Foundations. *Measure Algebras and Applications*, 3(1), 43–54. http://doi.org/10.22091/maa.2026.14588.1039

Publisher: University of Qom
Doi:http://doi.org/10.22091/maa.2026.14588.1039

1. Introduction

The theory of classical topological dynamical systems (TDS) is rooted in the deterministic framework of classical mechanics. One considers a phase space M, typically a manifold, and a homeomorphism $f : M \to M$ (or a flow φ_t) describing the time evolution. The dynamics of a definite initial state $x_0 \in M$ is given by the sequence $\{f^n(x_0)\}_{n\in\mathbb{Z}}$.

In stark contrast, quantum mechanics fundamentally replaces the notion of a definite initial state with a probabilistic description. The state of a system is not a point x_0 but a wave function ψ, and the observable reality is encoded in a probability density $\rho = |\psi|^2$. This foundational shift from certainty to probability necessitates a new dynamical framework at the level of densities. This paper aims to develop the mathematical foundations of such a framework, which we call *Quantum Topological Dynamical Systems* (QTDS), by systematically lifting classical dynamics to the space of probability measures.

1.1. Literature Review and Motivation. The interplay between classical and quantum dynamics has been a central theme in mathematical physics for decades. The Koopman–von Neumann approach [9, 15] provides an operator-theoretic formulation of classical mechanics, while the Perron–Frobenius operator [3, 11] describes the evolution of probability densities under classical dynamical systems. Our work bridges these classical approaches with quantum mechanical concepts.

In quantum mechanics, the evolution of wave functions follows the Schrödinger equation, and the corresponding probability densities satisfy the continuity equation [7, 10]. The Madelung transformation [12] reveals the hydrodynamic formulation of quantum mechanics, where the continuity equation plays a fundamental role. Our framework extends these ideas to a general topological setting.

Recent work in operator algebras and non-commutative geometry [5] has explored quantum dynamical systems from an algebraic perspective. However, our approach differs by maintaining the classical manifold structure while quantizing the state space from points to probability densities. This perspective is closer in spirit to quantum measurement theory [4] and the statistical mechanics of quantum systems [14].

The concept of microscopic fixed points introduced in this paper has precursors in the study of invariant measures and stationary distributions [16], but our definition provides a pointwise characterization that is novel in the literature. The cocycle conditions we employ are standard in the theory of random dynamical systems [1] and appear naturally in thermodynamic formalism [13] and transfer operator theory [3], though our focus is on the topological rather than statistical properties.

1.2. Main Contributions. In a QTDS, the fundamental object is not a map or a flow on M, but a map F or a flow Φ_t on the space of continuous probability densities $\mathcal{D}(M)$. Our primary goal is to reinterpret and generalize the core concepts of TDS within this new paradigm.

A central concept in TDS is that of a fixed point: a state x_0 such that $f(x_0) = x_0$. The naive quantum analogue would be a fixed density ρ_0 such that $F(\rho_0) = \rho_0$ or $\Phi_t(\rho_0) = \rho_0$. However, this is often too restrictive. We are frequently interested in points in the manifold M itself. Therefore, we define a point $x_0 \in M$ to be a *microscopic fixed point* if for every initial density ρ_0 and for all times t, the evolved density $F(\rho_0)$ or $\Phi_t(\rho_0)$ assigns the same probability to x_0 as the initial density did:

$$F(\rho_0)(x_0) = \rho_0(x_0)$$

or

$$\Phi_t(\rho_0)(x_0) = \rho_0(x_0).$$

This means the probability of finding the system at x_0 is constant in time. In this paper, we will rigorously define QTDS, provide canonical examples, and systematically redefine and prove analogues of classical results, starting with fixed and periodic points.

1.3. Justification of Terminology and Scope. The term *Quantum* in our title requires clarification. We use it in a specific, formal sense to indicate a framework whose axioms are modeled on foundational aspects of quantum mechanics, rather than to describe a system with genuine quantum (non-commutative) observables.

Philosophical Stance and Restrictive Focus: It is crucial to emphasize that the primary goal of this foundational paper is *not* to develop the most general possible theory of density evolution. Rather, we intentionally restrict our attention to quantum dynamics of the specific form given in Equations (2.3) and (2.6)—that is, weighted lifts of a classical dynamics (f, φ_t) via a cocycle h.

This restriction serves a deliberate purpose: to use the cocycle h (with generator σ) as a controlled *"microscope"* or *probabilistic lens* through which to study the classical system. The classical flow φ_t is not just an example; it is the *essential underlying object* of our study. The framework is designed to translate classical topological concepts (fixed points, periodic points) into probabilistic language *while preserving their relationship to the deterministic trajectory structure.*

The question of whether more general density evolutions (e.g., diffusive processes as in the heat equation) can be accommodated within the definition is mathematically valid. Indeed, Definition 2.2 is broad enough to include them. However, including such examples would dilute the paper's core mission: to establish a precise dictionary between classical deterministic properties and their probabilistic shadows *within a tightly coupled formalism.* The weighted lift construction guarantees this coupling.

Thus, while the referee's suggestion to explore non-lift examples points to an interesting generalization, it falls outside the self-imposed scope of this foundational work. Our aim is to build a coherent theory where the classical dynamics remains the reference point, and the "quantum" aspect enters solely through the probabilistic state representation and the multiplicative cocycle weight.

Our framework is "quantum" in the following precise ways:

(1) **State Space:** The fundamental object is a probability density $\rho \in \mathcal{D}(M)$, analogous to the position probability density $|\psi|^2$ in quantum mechanics, not a point in phase space.
(2) **Dynamical Law:** The evolution is defined directly on this space of densities ($F : \mathcal{D}(M) \to \mathcal{D}(M)$), mirroring the Schrödinger picture's evolution of states.
(3) **Key Equation:** A primary example and motivation is the continuity equation (2.9), which is exactly the equation governing probability conservation in the hydrodynamic (Madelung) formulation of quantum mechanics.

However, our theory is *classical* in these crucial aspects:

(1) **Underlying Space:** The configuration space M is a classical manifold.
(2) **Observables:** There is no introduction of non-commuting operators; observables are classical functions on M.
(3) **Superposition & Interference:** The framework deals with classical probability densities, not complex amplitudes, and therefore does not exhibit quantum interference.

Thus, the present work is more accurately described as a *probabilistic topological dynamics* inspired by the quantum mechanical worldview. The terminology "QTDS" serves to emphasize this conceptual lineage and to distinguish it from classical TDS, which deals with point trajectories. The primary goal is to establish a rigorous mathematical structure that translates concepts like fixed points, periodic points, invariant sets and (in future work) ergodicity from the deterministic realm to the probabilistic one. This provides a new lens through which to analyze classical systems and creates a formal bridge to the probabilistic nature of quantum theory.

2. Definition and Examples of Quantum Dynamics

Throughout this paper, let M be a finite-dimensional, oriented Riemannian manifold. Let $\mathcal{D}(M)$ denote the space of continuous probability density functions on M, i.e., continuous functions $\rho : M \to [0, \infty)$ with $\int_M \rho = 1$.

Topological Structure: We equip $\mathcal{D}(M)$ with the topology of uniform convergence on compact sets (compact-open topology). This choice is natural as it ensures that pointwise evaluation $\rho \mapsto \rho(x)$ is continuous for each $x \in M$. All maps $F : \mathcal{D}(M) \to \mathcal{D}(M)$ and flows Φ_t considered in this paper are assumed to be *continuous* with respect to this topology unless otherwise stated. This continuity assumption is essential for a genuine *topological* dynamical systems theory.

Definition 2.1 (Discrete Quantum Dynamics). *A* discrete quantum dynamics *on M is a homeomorphism $F : \mathcal{D}(M) \to \mathcal{D}(M)$.*

Definition 2.2 (Continuous Quantum Dynamics). *A* continuous quantum dynamics *(or* quantum flow*) on M is a one-parameter family of homeomorphisms $\{\Phi_t : \mathcal{D}(M) \to \mathcal{D}(M)\}_{t\in\mathbb{R}}$ such that:*

(1) $\Phi_0 = Id$,
(2) $\Phi_s \circ \Phi_t = \Phi_{s+t}$ *for all* $s, t \in \mathbb{R}$.

More general spaces (e.g., L^1 densities) may be considered in future work, but continuity is natural for pointwise evaluations in our definitions of microscopic fixed points. Note that we do *not* assume linearity of F or Φ_t. While many classical examples (like the Perron-Frobenius operator) are linear, our framework also accommodates the nonlinear factor $\exp(h_t(x))$ in the weighted lift construction. The theory is developed at this level of generality.

A primary source of examples comes from lifting a classical dynamics to the space of densities.

Example 2.3 (Induced Quantum Dynamics from a Classical Map). *Let $f : M \to M$ be a homeomorphism. Define the discrete quantum dynamics $F : \mathcal{D}(M) \to \mathcal{D}(M)$ by*

$$F(\rho)(x) = \rho(f^{-1}(x)). \tag{2.1}$$

This is a homeomorphism, with its inverse given by $F^{-1}(\rho)(x) = \rho(f(x))$. The sequence $F_n = F^n$ satisfies $F_{n+m} = F_n \circ F_m$ and $\operatorname{supp}(F_n(\rho)) = f^n(\operatorname{supp}(\rho))$.

This construction is essentially the Perron-Frobenius operator [11] for the map f. Our framework generalizes this classical operator to include multiplicative cocycles.

A more general construction involves a multiplicative factor.

Example 2.4 (Weighted Quantum Dynamics). *Let $f : M \to M$ be a homeomorphism and let $\{h_n : M \to \mathbb{R}\}_{n\in\mathbb{Z}}$ be a sequence of continuous functions satisfying the* cocycle condition

$$h_{m+n}(x) = h_n(x) + h_m(f^{-n}(x)) \quad \text{for all } m, n \in \mathbb{Z}, x \in M. \tag{2.2}$$

Define the quantum dynamics $F_n : \mathcal{D}(M) \to \mathcal{D}(M)$ *by*

$$F_n(\rho)(x) = \rho(f^{-n}(x)) \exp(h_n(x)). \tag{2.3}$$

This defines a quantum dynamics, as the cocycle condition (2.2) *ensures the group property* $F_{n+m} = F_n \circ F_m$.

Proof. We verify the group property

$$\begin{aligned} F_n(F_m(\rho))(x) &= F_m(\rho)(f^{-n}(x)) \exp(h_n(x)) \\ &= \rho(f^{-m}(f^{-n}(x))) \exp(h_m(f^{-n}(x)) + h_n(x)) \\ &= \rho(f^{-(n+m)}(x)) \exp(h_{n+m}(x)) \quad \text{(by (2.2))} \\ &= F_{n+m}(\rho)(x). \end{aligned}$$

□

A canonical choice for the cocycle h_n is given by summing a potential function along the orbit.

Example 2.5. *Let* $\sigma : M \to \mathbb{R}$ *be a continuous function. Define*

$$h_n(x) = \sum_{k=0}^{n-1} \sigma(f^{k-n}(x)). \tag{2.4}$$

Proof. Case 1: $n, m > 0$. We compute

$$\begin{aligned} h_{n+m}(x) &= \sum_{k=0}^{n+m-1} \sigma(f^{k-(n+m)}(x)) \\ &= \sum_{k=0}^{n-1} \sigma(f^{k-(n+m)}(x)) + \sum_{k=n}^{n+m-1} \sigma(f^{k-(n+m)}(x)). \end{aligned}$$

In the second sum, let $j = k - n$. Then j runs from 0 to $m-1$, and

$$f^{k-(n+m)}(x) = f^{j-m}(x) = f^{j-m}(f^{-n}(f^n(x))) = f^{-m+j}(f^{-n}(x)).$$

Thus

$$h_{n+m}(x) = \sum_{k=0}^{n-1} \sigma(f^{k-n}(f^{-m}(x))) + \sum_{j=0}^{m-1} \sigma(f^{j-m}(f^{-n}(x))).$$

Now observe

$$\begin{aligned} h_n(x) &= \sum_{k=0}^{n-1} \sigma(f^{k-n}(x)), \\ h_m(f^{-n}(x)) &= \sum_{j=0}^{m-1} \sigma(f^{j-m}(f^{-n}(x))). \end{aligned}$$

Therefore, $h_{n+m}(x) = h_n(x) + h_m(f^{-n}(x))$ for $n, m > 0$.

Case 2: General integer n, m. For negative indices, we extend the definition consistently via the cocycle condition. For $n < 0$, define

$$h_n(x) = -h_{-n}(f^n(x)).$$

This ensures the cocycle property holds for all integers by the group homomorphism property. One may verify directly for each sign combination that the identity is satisfied.

Since the cocycle condition holds for positive indices and the extension to negative indices is defined to preserve the condition, the result follows for all $m, n \in \mathbb{Z}$. □

The continuous-time analogues of these examples are as follows.

Example 2.6 (Continuous Quantum Flow)**.** *Let* $\{\varphi_t : M \to M\}_{t\in\mathbb{R}}$ *be a flow (a one-parameter group of homeomorphisms). Let* $\{h_t : M \to \mathbb{R}\}_{t\in\mathbb{R}}$ *be a family of functions satisfying the continuous cocycle condition:*

$$h_{s+t}(x) = h_t(x) + h_s(\varphi_{-t}(x)) \quad \text{for all } s,t \in \mathbb{R}, x \in M. \tag{2.5}$$

Then the map $\Phi_t : \mathcal{D}(M) \to \mathcal{D}(M)$ *defined by*

$$\Phi_t(\rho)(x) = \rho(\varphi_{-t}(x)) \exp(h_t(x)) \tag{2.6}$$

is a continuous quantum flow. Furthermore, $\mathrm{supp}(\Phi_t(\rho)) = \varphi_t(\mathrm{supp}(\rho))$*.*

Example 2.7. *Let* $\sigma : M \to \mathbb{R}$ *be continuous. The function*

$$h_t(x) = \int_0^t \sigma(\varphi_{s-t}(x))\mathrm{d}s \tag{2.7}$$

satisfies the cocycle condition (2.5)*.*

Proof. Let $h_t(x) = \int_0^t \sigma(\varphi_{s-t}(x))ds$. For any $s,t \in \mathbb{R}$ and $x \in M$, we compute

$$\begin{aligned} h_{s+t}(x) &= \int_0^{s+t} \sigma(\varphi_{u-(s+t)}(x))du \\ &= \int_{-s}^{t} \sigma(\varphi_{r-t}(x))dr \quad \text{(substitute } r = u - s\text{)} \\ &= \int_{-s}^{0} \sigma(\varphi_{r-t}(x))dr + \int_0^t \sigma(\varphi_{r-t}(x))dr. \end{aligned}$$

The second term equals $h_t(x)$. For the first term, substitute $v = -r$

$$\begin{aligned} \int_{-s}^{0} \sigma(\varphi_{r-t}(x))dr &= \int_s^0 \sigma(\varphi_{-v-t}(x))(-dv) \\ &= \int_0^s \sigma(\varphi_{-v-t}(x))dv \\ &= \int_0^s \sigma(\varphi_{-v}(\varphi_{-t}(x)))dv \quad \text{(flow property)} \\ &= h_s(\varphi_{-t}(x)). \end{aligned}$$

Thus $h_{s+t}(x) = h_t(x) + h_s(\varphi_{-t}(x))$, verifying the cocycle condition. □

A physically crucial example arises from the continuity equation.

Theorem 2.8. *Let* V *be a smooth vector field on* M *with flow* φ_t*. Let* $\sigma = -\nabla \cdot V$*, where* $\nabla\cdot$ *is the divergence operator on* M *with respect to its Riemannian volume form. Define the quantum flow* Φ_t *by* (2.6) *with* h_t *given by* (2.7)*, i.e.,*

$$\Phi_t(\rho)(x) = \rho(\varphi_{-t}(x)) \exp\left(-\int_0^t (\nabla \cdot V)(\varphi_{s-t}(x))\mathrm{d}s\right). \tag{2.8}$$

For a given initial density ρ_0*, define* $\rho_t(x) = \Phi_t(\rho_0)(x)$*. Then* ρ_t *satisfies the continuity equation:*

$$\frac{\partial \rho_t}{\partial t} + \nabla \cdot (\rho_t V) = 0. \tag{2.9}$$

Conversely, if ρ_t *is a solution to* (2.9) *with initial condition* ρ_0*, then* ρ_t *is given by the formula* (2.8)*.*

Proof. Let $J(t,x) = \det(d\varphi_t(x))$ be the Jacobian determinant of the flow. By Liouville's formula [2]:

$$\frac{\partial}{\partial t}J(t,x) = (\nabla \cdot V)(\varphi_t(x))J(t,x), \quad J(0,x) = 1.$$

Solving this ODE gives:

$$J(t,x) = \exp\left(\int_0^t (\nabla \cdot V)(\varphi_s(x))ds\right).$$

The pushforward density $(\varphi_t)_*\rho_0$ is given by [8]:

$$(\varphi_t)_*\rho_0(x) = \rho_0(\varphi_{-t}(x))J(-t,x)^{-1}.$$

Substituting the formula for $J(-t,x)$:

$$(\varphi_t)_*\rho_0(x) = \rho_0(\varphi_{-t}(x)) \exp\left(-\int_0^{-t} (\nabla \cdot V)(\varphi_s(x))ds\right).$$

Changing variables $s \mapsto -s$ in the integral yields exactly (2.8).

Equivalently, the pushforward density satisfies $\frac{d}{dt}(\varphi_t)_*\rho_0 = -\mathcal{L}_V((\varphi_t)_*\rho_0)$, where $\mathcal{L}_V$ is the Lie derivative along V. This is the geometric formulation of the continuity equation (2.9).

Since the pushforward density $(\varphi_t)_*\rho_0$ classically satisfies the continuity equation [6], and our formula equals this pushforward, ρ_t satisfies (2.9).

Conversely, by uniqueness for the continuity equation [6], any solution must equal the pushforward density, hence is given by (2.8). □

Remark 2.9. *For the quantum flow (2.8) we have:*

(a) $\varphi_t(\text{supp}(\rho_0)) = \text{supp}(\rho_t)$.
(b) If ρ_0 is a probability density ($\int_M \rho_0 = 1$), then $\Phi_t(\rho_0)$ is also a probability density for all t.

2.1. Existence, Uniqueness, and Generators. A fundamental question in any dynamical theory is: given an infinitesimal prescription, does a global flow exist? For classical flows generated by vector fields, the answer is provided by the theory of ordinary differential equations. For quantum flows on $\mathcal{D}(M)$, we can formulate an analogous theory.

Definition 2.10 (Infinitesimal Generator)**.** *Let $\{\Phi_t\}$ be a continuous quantum flow on $\mathcal{D}(M)$. Its* infinitesimal generator *is the operator $\mathcal{L}$ defined on a dense subspace of $\mathcal{D}(M)$ by*

$$\mathcal{L}\rho = \lim_{t\to 0} \frac{\Phi_t(\rho) - \rho}{t},$$

whenever the limit exists in the topology of $\mathcal{D}(M)$.

The primary example motivating our work is the continuity equation flow from Theorem 2.8. In that case, the generator is the first-order differential operator:

$$\mathcal{L}\rho = -\nabla \cdot (\rho V).$$

More generally, one could consider generators of the form $\mathcal{L}\rho = -V\cdot\nabla\rho + \sigma\rho$, where $\sigma : M \to \mathbb{R}$ is a continuous function. This corresponds to the weighted continuity equation

$$\frac{\partial \rho_t}{\partial t} + V \cdot \nabla\rho = \sigma\rho,$$

whose solution is precisely the quantum flow (2.6) with h_t given by (2.7).

Theorem 2.11 (Local Existence and Uniqueness for Lipschitz Generators). *Let $\mathcal{L} : \mathcal{D}(M) \to \mathcal{D}(M)$ be a (possibly nonlinear) operator that is Lipschitz continuous on bounded subsets of $\mathcal{D}(M)$. Then for any initial $\rho_0 \in \mathcal{D}(M)$, there exists a unique local quantum flow Φ_t defined for t in some interval $(-\epsilon, \epsilon)$ such that $\frac{d}{dt}\Phi_t(\rho_0) = \mathcal{L}(\Phi_t(\rho_0))$.*

Sketch. The proof follows standard arguments from the theory of ordinary differential equations on Banach spaces, applied to the space of continuous functions on M. The Lipschitz condition ensures local existence and uniqueness via the Picard-Lindelöf theorem in this function space setting. □

This theorem provides a general mechanism for constructing quantum flows from generators. However, the specific weighted lift construction (2.6) remains our primary focus, as it maintains a direct correspondence with an underlying classical flow φ_t.

2.2. Relation to Koopman and Transfer Operators. Our framework has natural connections to two classical operator-theoretic approaches to dynamics:

(1) **Koopman Operator:** For a classical map $f : M \to M$, the Koopman operator $U : L^\infty(M) \to L^\infty(M)$ acts on observables by $Ug(x) = g(f(x))$. It is the dual to the Perron-Frobenius operator acting on densities. Our weighted quantum dynamics (2.3) can be viewed as a *nonlinear* version of the Perron-Frobenius operator, where the multiplicative factor $\exp(h_n(x))$ introduces state-dependent weighting.

(2) **Transfer Operators:** In thermodynamic formalism, transfer (Ruelle-Perron-Frobenius) operators of the form

$$\mathcal{L}_\sigma g(x) = \sum_{f(y)=x} e^{\sigma(y)} g(y)$$

play a central role. Our discrete weighted dynamics $F(\rho)(x) = \rho(f^{-1}(x)) \exp(\sigma(f^{-1}(x)))$ is essentially the deterministic version of this construction where f is invertible. The cocycle condition (2.2) ensures the group property that is automatic for powers of a fixed transfer operator.

A fundamental question raised by the referee is: Is every "reasonable" quantum dynamics F of the weighted lift form (2.3)? For dynamics that *preserve the support in a deterministic way*—i.e., for which there exists a map $f : M \to M$ such that $\mathrm{supp}(F(\rho)) = f(\mathrm{supp}(\rho))$ for all ρ—one can show that F must be of the form $F(\rho)(x) = \rho(f^{-1}(x))w(x)$ for some weight function w. The cocycle condition on w (or its time-dependent analogue) then emerges from requiring the group property. Thus, within the class of support-preserving dynamics, our formulation is essentially complete.

However, as noted in Section 1.3, this paper intentionally restricts to this class to maintain a tight connection with classical dynamics. More general quantum dynamics that allow support mixing or diffusion (like the heat equation) are mathematically possible within Definition 2.2 but lie outside our current focus.

2.3. Physical Relevance: Madelung Transform and Quantum Mechanics. The continuity equation (2.9) is fundamental in physics and provides a direct bridge to quantum mechanics via the Madelung transform [12]. Consider the Schrödinger equation for a wave function $\psi : M \times \mathbb{R} \to \mathbb{C}$

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + U\psi,$$

where $\hbar$ is Planck's constant, m is mass, and U is the potential energy.

Using the polar decomposition $\psi = R\exp(iS/\hbar)$ with R, S real-valued, we set

$$\rho = |\psi|^2 = R^2,$$
$$V = \frac{1}{m}\nabla S.$$

The Schrödinger equation then implies that ρ and V satisfy

(1) The continuity equation: $\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho V) = 0$.
(2) A modified Hamilton-Jacobi equation (which involves the quantum potential).

Thus, the probability density ρ for a quantum particle evolves exactly according to a QTDS of the form (2.8) with $\sigma = -\nabla\cdot V$. This provides a concrete physical realization of our framework: the quantum mechanical probability density flow is a special case of a quantum topological dynamical system.

Importantly, in this quantum setting, the vector field $V = \nabla S/m$ is generally not determined by the classical equations of motion but contains additional quantum corrections. This illustrates how QTDS can model dynamics beyond classical deterministic flows while maintaining a clear probabilistic structure.

3. Microscopic Fixed Points

We now introduce the quantum analogue of a fixed point.

Definition 3.1 (Microscopic Fixed Point)**.** *Let F be a quantum dynamics on M.*

(a) In the discrete case, a point $x_0 \in M$ is a microscopic fixed point *if for all $\rho \in \mathcal{D}(M)$ we have $F(\rho)(x_0) = \rho(x_0)$.*
(b) In the continuous case, a point $x_0 \in M$ is a microscopic fixed point *if for all $\rho \in \mathcal{D}(M)$ and all $t \in \mathbb{R}$, $\Phi_t(\rho)(x_0) = \rho(x_0)$.*

This definition captures the idea that the probability of finding the system at x_0 is invariant under the time evolution, regardless of the initial state.

Theorem 3.2. *Let $f : M \to M$ be a homeomorphism and let F_n be the quantum dynamics of the form (2.3), where $\{h_n\}$ satisfies the cocycle condition (2.2).*

(a) If $x_0 \in M$ is a microscopic fixed point for F, then x_0 is a fixed point for f (i.e., $f(x_0) = x_0$).
(b) Suppose x_0 is a fixed point for f. Then x_0 is a microscopic fixed point for F if and only if $h_n(x_0) = 0$ for all $n \in \mathbb{Z}$.
(c) In particular, if h_n is given by (2.4), then a fixed point x_0 of f is a microscopic fixed point if and only if $\sigma(x_0) = 0$.

Proof. (a) Assume x_0 is a microscopic fixed point. We will show that $f^{-1}(x_0) = x_0$, which implies $f(x_0) = x_0$. Suppose, for contradiction, that $f^{-1}(x_0) \neq x_0$. Choose a density ρ such that $\rho(x_0) > 0$ but $\rho(f^{-1}(x_0)) = 0$. Then

$$F_1(\rho)(x_0) = \rho(f^{-1}(x_0))\exp(h_1(x_0)) = 0.$$

But since x_0 is a microscopic fixed point, $F_1(\rho)(x_0) = \rho(x_0) > 0$, a contradiction. Hence, $f^{-1}(x_0) = x_0$, meaning $f(x_0) = x_0$.

Note: The same argument works for any n by choosing ρ with $\rho(x_0) > 0$ and $\rho(f^{-n}(x_0)) = 0$. Since microscopic fixed point condition holds for all n, we conclude $f^{-n}(x_0) = x_0$ for all n, and in particular for $n = 1$.

(b) Suppose $f(x_0) = x_0$. Then $f^{-n}(x_0) = x_0$ for all n. The condition for x_0 to be a microscopic fixed point is

$$F_n(\rho)(x_0) = \rho(x_0)\exp(h_n(x_0)) = \rho(x_0) \quad \text{for all } \rho.$$

This holds if and only if $\exp(h_n(x_0)) = 1$, i.e., $h_n(x_0) = 0$.

(c) If h_n is given by (2.4) and $f(x_0) = x_0$, then $h_n(x_0) = n\sigma(x_0)$. Thus, $h_n(x_0) = 0$ for all n if and only if $\sigma(x_0) = 0$. □

An analogous theorem holds in the continuous setting.

Theorem 3.3. *Let φ_t be a flow on M and Φ_t a quantum flow of the form (2.6), where $\{h_t\}$ satisfies (2.5).*

(a) If x_0 is a microscopic fixed point for Φ_t, then x_0 is a fixed point for φ_t (i.e., $\varphi_t(x_0) = x_0$ for all t).

(b) If x_0 is a fixed point for φ_t, then x_0 is a microscopic fixed point for Φ_t if and only if $h_t(x_0) = 0$ for all t.

(c) If h_t is given by (2.7), then a fixed point x_0 is microscopic if and only if $\sigma(x_0) = 0$. If $\sigma = -\nabla \cdot V$, this condition is $\nabla \cdot V(x_0) = 0$.

4. Microscopic Periodic Points

We now generalize the concept of a fixed point to that of a periodic point.

Definition 4.1 (Microscopic Periodic Point)**.** *Let F be a quantum dynamics on M.*

(a) In the discrete case, a point $x_0 \in M$ is microscopic periodic *with period $\alpha \in \mathbb{Z}^+$ if for all $\rho \in \mathcal{D}(M)$, $F_\alpha(\rho)(x_0) = \rho(x_0)$.*

(b) In the continuous case, a point $x_0 \in M$ is microscopic periodic *with period $\alpha > 0$ if for all $\rho \in \mathcal{D}(M)$, $\Phi_\alpha(\rho)(x_0) = \rho(x_0)$.*

Theorem 4.2. *Let $f : M \to M$ be a homeomorphism and F_n a quantum dynamics of the form (2.3).*

(a) If x_0 is microscopic periodic with period α, then x_0 is classically periodic with period α (i.e., $f^\alpha(x_0) = x_0$).

(b) If x_0 is classically periodic with period α, then x_0 is microscopic periodic with period α if and only if $h_\alpha(x_0) = 0$.

(c) If h_n is given by (2.4), then a classical α-periodic point x_0 is microscopic α-periodic if and only if $\sum_{k=0}^{\alpha-1} \sigma(f^k(x_0)) = 0$.

Proof. (a) The proof is similar to Theorem 3.2(a). Assume x_0 is microscopic α-periodic. If $f^{-\alpha}(x_0) \neq x_0$, choose ρ with $\rho(x_0) > 0$ and $\rho(f^{-\alpha}(x_0)) = 0$. This leads to the contradiction $0 = F_\alpha(\rho)(x_0) = \rho(x_0) > 0$. Thus $f^{-\alpha}(x_0) = x_0$, so $f^\alpha(x_0) = x_0$.

(b) If $f^\alpha(x_0) = x_0$, then $f^{-\alpha}(x_0) = x_0$. The microscopic periodicity condition becomes

$$F_\alpha(\rho)(x_0) = \rho(x_0)\exp(h_\alpha(x_0)) = \rho(x_0),$$

which holds for all ρ if and only if $\exp(h_\alpha(x_0)) = 1$, i.e., $h_\alpha(x_0) = 0$.

(c) If h_n is given by (2.4) and $f^\alpha(x_0) = x_0$, then

$$h_\alpha(x_0) = \sum_{k=0}^{\alpha-1} \sigma(f^{k-\alpha}(x_0)) = \sum_{k=0}^{\alpha-1} \sigma(f^k(x_0)).$$

The result follows from part (b). □

The set of microscopic periodic points shares a key property with its classical counterpart.

Theorem 4.3. *Let $f : M \to M$ and F_n be as in Theorem 4.2. If x_0 is microscopic periodic with period α, then for every $n \in \mathbb{Z}$, the point $f^n(x_0)$ is also microscopic periodic with period α.*

Proof. By Theorem 4.2(a), x_0 is classically periodic: $f^\alpha(x_0) = x_0$. It follows that $f^\alpha(f^n(x_0)) = f^n(f^\alpha(x_0)) = f^n(x_0)$, so $f^n(x_0)$ is classically periodic. We must show $h_\alpha(f^n(x_0)) = 0$.

For all n, we have $h_{\alpha-n}(x_0) = h_\alpha(x_0) + h_{-n}(f^{-\alpha}(x_0)) = h_{-n}(x_0)$. On the other hand $h_{\alpha-n}(x_0) = h_{-n}(x_0) + h_\alpha(f^n(x))$. Hence, $h_\alpha(f^n(x)) = 0$. □

Remark 4.4. *The concepts of microscopic fixed and periodic points naturally extend to sets.*

(a) *Let Φ_t be a quantum flow on M. We call a measurable set $A \subseteq M$* microscopically invariant *if for all $\rho \in \mathcal{D}(M)$ and all $t \in \mathbb{R}$,*

$$\int_A \Phi_t(\rho)(x)\,dx = \int_A \rho(x)\,dx.$$

(b) *In the discrete case, a set A is microscopically invariant under a quantum dynamics F if for all $\rho \in \mathcal{D}(M)$,*

$$\int_A F(\rho)(x)\,dx = \int_A \rho(x)\,dx.$$

This definition captures the intuitive notion that a set is invariant if the total probability mass within it is conserved under the dynamics. Microscopic invariant sets represent a novel concept that interpolates between measure-theoretic invariance (conservation of probability mass) and topological invariance (set-wise preservation by the flow). They provide a rich field for further investigation within the QTDS framework.

5. Conclusion and Outlook

In this paper, we have laid the foundation for the theory of Quantum Topological Dynamical Systems (QTDS)—a probabilistic, quantum-inspired reformulation of dynamics on the space of probability densities. We provided a general mechanism for constructing such dynamics from classical systems and introduced the pivotal concepts of microscopic fixed points, periodic points, and invariant sets. We established precise relationships between these new quantum notions and their classical precursors.

Beyond these core results, we have addressed key foundational aspects: (1) specifying the topological structure on $\mathcal{D}(M)$ and continuity requirements, (2) discussing existence and uniqueness via infinitesimal generators, (3) clarifying the relationship to classical operator-theoretic approaches (Koopman and transfer operators), and (4) initializing the theory of microscopic invariant sets. These elements justify the ”Foundations” title and provide a solid basis for future development.

The framework, while intentionally restricted to weighted lifts of classical flows, establishes a precise dictionary for translating classical topological concepts into the language of evolving probability densities. This provides a novel probabilistic lens through which to analyze deterministic dynamical systems.

Specific Open Questions and Future Directions:

(1) **Topological Refinement:** The next natural step is to topologize $\mathcal{D}(M)$ more carefully (e.g., with Wasserstein metrics or weak-* topology) and study the continuity properties of F and Φ_t in these topologies.
(2) **Ergodic Theory:** A microscopic version of ergodicity, mixing, and the decomposition theorem for invariant measures should be developed.

(3) **Non-commutative Extension:** Can the framework be extended to non-commutative spaces (the state space of operator algebras) while preserving the dictionary between microscopic and classical concepts?

The relationship between our microscopic fixed points and the classical concept of invariant measures [16] has been partially elucidated through the theory of microscopic invariant sets. Additionally, the connection with transfer operator theory [3] and thermodynamic formalism [13] suggests potential applications in statistical mechanics and dynamical systems theory.

Funding: This research was in part supported by grants from IPM Schools of Mathematics (No.1404570114) and Physics (IPM/P-2024/52), Iran for the second author.

References

[1] Arnold, L. (1998). Random Dynamical Systems. *Springer*. DOI: https://doi.org/10.1007/978-3-662-12878-7.

[2] Arnold, V.I. (2013). Mathematical methods of classical mechanics (2nd ed.). *Springer*. DOI: https://doi.org/10.1007/978-1-4757-2063-1.

[3] Baladi, V. (2000). Positive transfer operators and decay of correlations. *World Scientific*. DOI: https://doi.org/10.1142/3657.

[4] Busch, P., Lahti, P., & Mittelstaedt, P. (1996). The quantum theory of measurement (2nd ed.). *Springer*. ISBN 3540613552.

[5] Connes, A. (1994). Noncommutative geometry. *Academic Press*. ISBN: 978-0-12-185860-5.

[6] Evans, L.C. (2022). Partial differential equations (2nd ed.). *American Mathematical Society*.

[7] Griffiths, D.J., & Schroeter, D.F. (2018). Introduction to quantum mechanics (3rd ed.). *Cambridge University Press*. DOI: https://doi.org/10.1017/9781316995433.

[8] Katok, A., & Hasselblatt, B. (1995). Introduction to the modern theory of dynamical systems. *Cambridge University Press*. DOI: https://doi.org/10.1017/CBO9780511809187.

[9] Koopman, B.O. (1931). Hamiltonian systems and transformation in Hilbert space. *Proceedings of the National Academy of Sciences*, 17(5), 315–318. DOI: https://doi.org/10.1073/pnas.17.5.315.

[10] Landau, L.D., & Lifshitz, E.M. (2013). Quantum mechanics: non-relativistic theory (3rd ed.). *Elsevier*.

[11] Lasota, A., & Mackey, M.C. (2013). Chaos, fractals, and noise: stochastic aspects of dynamics (2nd ed.). *Springer*. ISBN-10. 0387940499.

[12] Madelung, E. (1927). Quantentheorie in hydrodynamischer Form. *Zeitschrift für Physik*, 40(3), 322–326. DOI: https://doi.org/10.1007/BF01400372.

[13] Ruelle, D. (2010). Thermodynamic formalism: the mathematical structure of equilibrium statistical mechanics (2nd ed.). *Cambridge University Press*. DOI: https://doi.org/10.1017/CBO9780511617546.

[14] Thirring, W. (2002). Quantum mathematical physics: atoms, molecules and large systems (2nd ed.). *Springer*. DOI: https://doi.org/10.1007/978-3-662-05008-8.

[15] von Neumann, J. (1996). Mathematische Grundlagen der Quantenmechanik. *Springer*. DOI: https://doi.org/10.1007/978-3-642-61409-5.

[16] Walters, P. (2000). An introduction to ergodic theory (2nd ed.). *Springer*. ISBN 0387951520.